A Monograph for the Teaching of Geometry and Measurement in Initial Teacher Education in South Africa: Foundation and Intermediate Phases

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Abstract
When learning Geometry, students may tend to memorise properties, relationships, and formulae and may even attempt to solve problems mechanically. However, Geometry provides students with opportunities to engage with logic and reasoning beyond only symbols, but within spatial contexts as well. There is a tendency that Geometry, as it stands in the curriculum, is interpreted and subsequently taught as a list of separate, unrelated bullet points. Secondly, Geometric elements tend to be memorised according to their appearance, or their definitions are simply memorised without understanding. This qualitative case study underpinned by the Van Hiele theory aimed to explore the "big ideas" that permeate the effective teaching of Geometry, and how these "big ideas" promote an understanding of the connectedness between concepts in Geometry, Measurement, Number, and in the environment. Data were collected from 15 mathematics teacher educators across 10 Higher Education Institutions in South Africa that participated in a Primary Teacher Education project, which focused on developing new teacher graduates' ability to teach Geometry and Measurement. Data was collected via document analysis, questionnaires and focus group interviews. The study found that geometrical properties, measurement, transformations, invariance and visualization are the big ideas that permeates the teaching of geometry and measurement. These "big ideas" has the potential to influence how mathematics teacher educators re-organise and sequence their teaching and learning activities on geometry in pre-service mathematics teacher education curricula in connected ways. Furthermore, pre-service mathematics teachers must consider the 'big ideas' in the design of their lessons for work-integrated learning practices.

Keywords: Big Ideas, Geometrical Properties, Measurement, Transformations, Invariance and Visualization

Introduction

Both teaching and learning experience indicate that Geometry remains a neglected part of Mathematics within the school curriculum. Lappan (1999, p. 1) refers to Geometry as the "forgotten Strand." Singh (2017, p. 633) maintains that Geometry "is often avoided in the syllabus." Piper, Ralaigita, Akach and King (2016) talk of an "insufficient" way in which sub-Saharan countries have been teaching Mathematics in general, and Geometry specifically.

In the absence of the current curriculum pacesetters, Geometry was often relegated to being taught during part of the last school term each year. It might have been that privilege was afforded to numbers and operations, because of a conception that numbers are synonymous with Mathematics, or because teachers did not feel comfortable teaching Geometry. We suspect that it was, and still is, the latter (Lappan, 1999).

This study does not intend to propose Geometry (identified as Space and Shape, and Measurement in the General Education and Training Band of South African schooling) to be more important than Numbers, although it does seem to be the stepchild of school mathematics. It can be argued, however, that these two fields of knowledge, *numbers* and *space* form the foundational structure of Mathematics as it ought to be taught in schools. Liping Ma captures the essence of school mathematics with her statement: *"Mathematics is an area of science that concerns spatial and numeric relationships in which reasoning is based on these relationships"* (Ma, 1999).

Geometry was a voluntary content area in the senior certificate examinations (Grade 12) for six years in South Africa between 2006 and 2012 (Ubah & Bansilal, 2019). This, in itself, could have been an admission or recognition that Geometry was not being taught adequately, and hence would impact negatively on senior certificate throughput statistics.

The PrimTED Project's Geometry and Measurement Working Group motivates the national and strategic importance of the proposed project by noting that:

> *"As evident across various sets of benchmarking tests and published literature, there seems to be challenge with the development of sound conceptual and procedural understanding of salient aspects of shape, space and measurement across primary years. Furthermore, the experience that education lecturers have at teaching aspects related to shape, shape and measurement at both under-graduate and postgraduate level has shown that even experienced teachers (who are well trained) have a backlog in their space, shape and measurement content knowledge as well as their pedagogical and curriculum specific knowledge of space, shape and measurement."* (Teaching & Learning Development Capacity Improvement Programme, Primary Teacher Education Project, Working Group 3 Brief, 2016)

This observation ties in with the motivation to research the causes of the "backlogs" in teachers' geometry content (van Putten, et al., 2010; citing Bowie, 2009) and pedagogical knowledge, and attempts to address this by exploring "big ideas" as a possible point of departure to develop the type of spatial reasoning required for effective engagement with Geometry at FET and tertiary education levels.

The mathematics content is developed for pre-service teachers who should essentially be prepared for Intermediate Phase teaching according to the requirements set out in MRTEQ (Minimum Requirements for Teacher Education Qualifications, 2019). "MRTEQ provides a basis for the construction of core curricula Initial Teacher Education (ITE) as well as for Continuing Professional Development (CPD) Programmes that accredited institutions must use in order to develop programmes leading to teacher education qualifications." (DHET, 2011, p. 6).

Initial meetings of the PrimTED Working Group 3 (Geometry and Measurement) decided that "big ideas" in teaching Geometry needed to be identified in order to focus the development of pre-service teacher training materials. Initially, consensus, between those at the initial forum did not exist as to what these "big ideas" needed to be. Suggestions included visualization, invariance, the Van Hiele Levels, and spatial reasoning as "big ideas" as well. It was argued that *spatial reasoning* should be a central idea in the teaching of Geometry at primary school level. Counter arguments reasoned that *spatial reasoning* was too broad an idea and might not incorporate geometric concepts as proposed within school mathematics. Subsequent discussion settled on *properties* in Geometry as being an important concept to consider, given especially that the manner in which the South African schools' curriculum was structured leaned towards the identification, comparison and description of objects and shapes. The identification, comparison and description of objects and shapes can be regarded as early engagement with properties in Geometry. Hence, *properties* became an initial "big idea" to be considered to guide the development of teaching materials for pre-service teachers.

It was at the point when thinking about how properties of geometric elements could be compared, either directly or indirectly, or eventually quantified, that the importance of *measurement* in the context of properties became apparent. Measurement already exists as a strand in mathematics within the South African schools' curriculum, and it was reasoned that *measurement*, as a proposed "big idea" would provide an opportunity to stress its interconnectedness with *properties as* related to the teaching of Geometry.

Armed with the innate ability (Feiberger, 2006; Palmer, 2011) to recognise sameness and difference in terms of form and size, considering position, perspective or orientation, accentuated to a level of mathematical accuracy by aspects of measurement, the idea of *transformations* became regarded as an additional "big idea" through the realization of invariance.

These "big ideas" provided three focal points on the Geometry content, and thus clear topics on which the sub-groups with the PrimTED Working Group 3 could work. The structure which was developed illustrated the interrelationships between the big ideas, and clarified that all three big ideas, as identified were interdependent. In other words, it was difficult to speak about any one "big idea" in the absence of any of the other two.

Cognizance had to be given to those core knowledges, such as knowledge of *position*, *direction* and *distance,* which learners possess prior to commencing school, in other words, prior to any exposure to an organized collection of content knowledge (the curriculum). This is particularly evident when consulting examples of curricula from several countries, where early concepts are based on innate knowledge, individual, pre-formalized schooling experience, and the observed environment (Izard, et al., 2011). Thus, *foundational knowledge* was acknowledged as a building block for the three "big ideas" of *properties*, *measurement*, and *transformations*.

Moreover, the resolution to distill the Geometry content within these three identified "big ideas" allowed for the maintenance of a big picture of Geometry initially, and then Mathematics as part of the broader perspective.

Rationale

In the face of a highly specific curriculum policy statement (CAPS), with its range of equally specific, tightly aligned teaching and learning resources, demonstrating an apparent lack of trust in teacher capacity, backed by the demands of numerous education departmental officials, in many geographic areas it seems that South African teachers have responded quite typically. Teachers, guided by bureaucratic structure and constant monitoring, obediently implement the curriculum, following the aligned textbooks and workbooks to the 'T'. This together with the pressures of classroom management and administrative tasks, may be transforming our South African teachers into *curriculum deliverers* rather than what they are supposed to be.

Boaler makes the point of how interconnected mathematical concepts are, unified by "a few really big and important ideas," but in contrast to what learners think that mathematics is, namely "a lot of different rules and methods" (Boaler, 2019).

It was this concern, highlighted in the previous paragraph and with the intent to encourage and maintain a broad perspective in the teaching of Geometry that the "big ideas" were envisaged. It is further argued that a focus on "big ideas" in teaching mathematics deepens teachers' subject knowledge and has the potential to promote or advance the development of relevant pedagogies, thereby emphasizing mathematical inter-connections (Barclay & Barnes, 2013).

To support this, wide research (Chi et al. 1982, p. 51) supports this argument in many other fields, with researchers consistently finding that experts, as opposed to novices, who operate off highly developed knowledge structures, which are more often than not organized around central concepts, or "big ideas" (Niemi et al., 2006).

Generally, then, the notion of "big ideas" stands in contrast to vast, detailed curriculum frameworks such as CAPS, yet provides an opportunity to distill the salient concepts within Geometry to allow for more effective teaching and learning.

Any curriculum structure, guided by carefully considered and selected "big ideas," should create a space for the desired way of thinking when learning Geometry in primary school classrooms, a way of reasoning that would support thinking for solving problems and justifying conjectures at high school level, as well as in tertiary education.

Aside from the issues of curriculum detail, Geometry in itself tends towards the presence and perhaps maintained, or even growing reliance of visual prompts and diagrammatic representations. And so, it should, as it busies itself with *space*. While diagrammatic prototypes serve their purpose in introducing shapes and objects, and representing relationships, there may exist the peril of engagement with geometric elements (in this case figures and forms) remaining at the most basic levels. That is, that learners may remain at the level of recognition, commenting or making decisions based on perception. This level being the lowest level as identified in the Van Hiele Levels of spatial reasoning (Mason, 2019).

At primary school level, it appears that Geometry is mainly about simply identifying figures and forms by their appearance, rather than their properties (Greenstein, 2014; Luneta, 2014). Apart from introducing more shapes or objects, no development in thinking about forms or figures is encouraged. Not much, if any spatial reasoning is developed in terms of learning Geometry. It is thus not surprising that learners are unable to make connections and use logic to solve problems that involve spatial aspects. This approach to learning Geometry is, Greenstein maintains, detrimental to young children's development in that their engagement with geometric concepts is not expanded beyond a set of conventional, rigid shapes, these shapes develop into a set of visual prototypes that could rule their thinking throughout their lives (Burger & Shaughnessy, 1986; Clements, 2004; Greenstein, 2014).

While diagrams and concrete objects may be unavoidable in the teaching of Geometry during the earlier years of schooling, the focus on what learners need to know, and more importantly *how they need to reason* when learning Geometry needs to be stressed.

As discussed above, a shift in focus from a broad curriculum focus towards "big ideas" in Geometry should lead to more flexible and generalizable application of knowledge, improved problems-solving and greater sense-making in the learning of Geometry (Niemi, et al., 2006).

Purpose of Study

Properties, as one of the "big ideas," presents an accessible point of departure for learning Geometry at the primary school level, and especially at the Foundation and Intermediate Phase levels. At this level of schooling, learners engage with objects and shapes as they commence a more formal relationship with Mathematics (reference). While objects and shapes – at this level of schooling – are regarded as independent, unrelated elements, learners are quite capable of distinguishing similarities and differences within groups of objects, or groups of shapes. This is in line with how Geometry is presented through the curriculum (CAPS), and thus it seems possible that a focus on *properties* as one of the "big ideas" presents ready access through the curriculum.

Mathematical engagement with the properties related to points, lines, shapes and objects, and how these can be accurately quantified, or transformed, requires more than recognition of prototypes, or memorization of properties, but the employment of spatial reasoning (Luneta, 2014), with visualisation, and the realization of the interconnectedness between these geometric elements.

The intent of this study then, is to describe how an approach, which focusses on spatial reasoning borne out of deep knowledge and understanding of geometric concepts can influence how pre-service teachers regard geometric elements (points, lines, figures and objects) as these are prescribed in the curriculum. The research targets pre-service teachers' spatial awareness, and resultant spatial reasoning, and reasons that this awareness is translated into effective teaching methodology. This logical "regard" for points, lines, figures and objects can be described and explained through the properties of these geometric elements.

If produced guidelines propose properties of shapes and objects, with an intent to promote spatial reasoning within the ambit of these properties, it cannot be guaranteed that readers, and eventual implementers of these texts will interpret these texts as they were intended. In short, interpretations may be superficial, and may result in the texts being memorised – only minimalistic as properties of objects and shapes. Therefore, in addition to highlighting properties in the teaching of objects and shapes, this study will reference *habits of the mind* in terms of how teachers can relate to the aforementioned properties and spatial relationships, as well as how these habits of the mind are initiated and developed from the *core knowledges* which are innate perceptions of our environment.

The question remains as to whether this regard for teaching and learning Geometry presents an effective approach for the development of future teachers.

Theoretical Framework

The theoretical framework underpinning this research consists of an amalgamation of various theories- Van Hiele model (Van de Walle et al, 2013); Realistic Mathematics Education (RME) (Freudenthal, 1991; Gravemeijer, 1994); relational vs instrumental learning (Skemp, 1976) to name but three- to ensure that the framework for geometric mathematics content development for pre-service student teachers is holistic by nature as opposed to viewing it as a set of unrelated and separate components. In addition, the use of various theories should ensure that a diversity of teaching and learning methods and strategies are incorporated to develop, package and present geometric content in ways that are readily accessible to students with diverse needs and competencies.

For the purpose of this paper the focus is mainly on the use and application of the Van Hiele levels of geometric development. The Van Hiele model is considered globally as essential in "designing and developing learning instruction to enhance students' higher order thinking skills in Geometry" (Naufal, Abdullah, Osman, Abu, and Ihsan, 2020; Atebe, and Schäfer, 2011). Likewise, Luneta (2014, p.74) is convinced that an understanding of the Van Hiele levels "enables teachers to identify the general direction of their students' learning and the level at which they are operating geometrically."

The proponents of the Van Hiele theory are husband and wife, Dinah Van Hiele-Geldof and Pierre Van Hiele. This theory is dualistic by nature in the sense that it consists of (1) levels of thinking and (2) phases of learning.

The van Hiele theory comprises five sequential and hierarchical discrete Levels of geometric thought specifically: Visualization, Analysis, Order (Informal Deduction), Deduction, and Rigour (Van Hiele, 1986; Armah and Kissi, 2019). Each of the said Levels defines the thought processes used in particular geometric related contexts. As learners advance from one Level to the next, the object of their geometric thinking changes (Armah and Kissi, 2019). At primary school level, learners will be inclined to transfer upward from level 1 to level 2. For example, at level 1, learners may identify shapes or geometrical objects by appearance only by comparing everyday objects, for example, 'it looks like a table top' or put the shape in a particular grouping or not (Armah & Kissi, 2019). Their language usage is basic (Vojkuvkova, 2012).

At level 2, learners begin analysing and naming properties of geometric shapes, however they may not yet grasp the interrelationship that exist between different categories of shapes such as rectangles and parallelograms. (Armah & Kissi, 2019). Then in Senior Phase level learners should advance to level 3 where they now have developed the ability to recognise the interrelationship between different types of different shapes for instance that a square as all the properties of a rectangle. Generally learners can come up with “meaningful definitions and give informal arguments to justify their reasoning at this Level” (Armah & Kissi, 2019, p. 3). As indicated earlier in the discussion the focus is on primary school teacher development and training thus the only the first three Van Hiele levels are considered relevant.

Teaching of geometry is structured into five phases of learning (Luneta, 2014), namely: information (familiarising learners with the geometry content and the pre-knowledge tested), directed orientation (learners are guided to uncover connections and identify content focus and engage with content), explication (learners verbalise their understandings of concepts), free orientation (learners complete complex tasks on their own), and integration (learners summarise what has been learned and create overviews of geometric concepts used) (Moru, Malebanye, Morobe, & George, 2021, pp. 20-21; Dongwi, 2014, p. 112).

The Van Hiele theory is not age-dependent. Hence the learners' progression from one level to the next is dependent on the effectiveness of the teaching and content acquisition opportunities that they are exposed to (Luneta, 2014; Robichaux-Davis & Guarino, 2016; Nisawa, 2018).

Objectives of Study

At the outset of this study, after in-depth discussions and planning, particular objectives were identified to ensure that the team remained focused and that the primary research aim was achieved. The said objectives are as follows:

1. To explore the “Big Ideas” for the teaching of Geometry and Measurement.
2. To develop a set of knowledge and practice standards for Geometry and Measurement for FP and IP.
3. To explore what Geometry and Measurement Content Knowledge should be included in mathematics teacher education curricula.
4. To explore what Geometry and Measurement Pedagogical Content Knowledge should be included in mathematics teacher education curricula.

Discussion

The discussion is intimately linked to Figure 1. This particular Figure illustrates how core knowledge and awareness of the natural and human-made environment provide a foundation for the realization of properties of geometric elements.

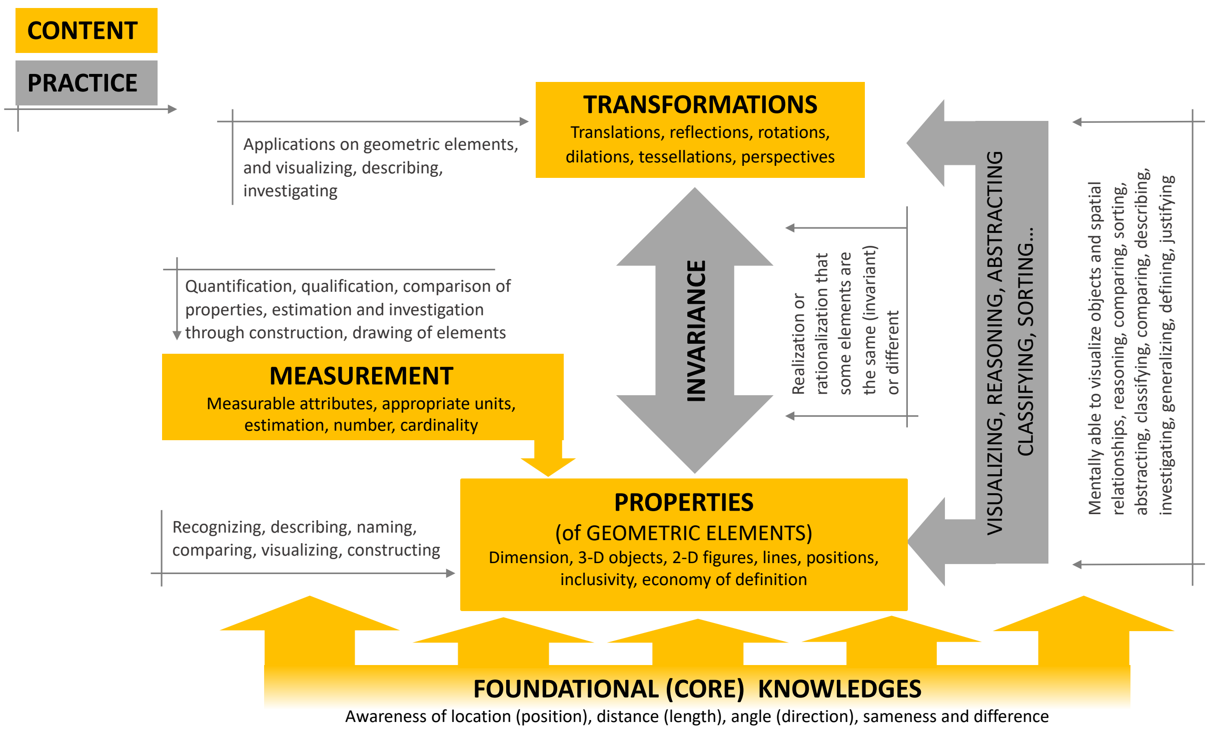


Figure 1: Conceptualising Geometry

Izard, Pica, Spelke, and Dehaene (Izard, et al. 2011)) conducted experiments with participants from an indigene group in the Amazon, the Munduruku (Feiberger, 2006; Palmer, 2011), as well as adults and age-matched children controls from the United States and France, and younger US children without education in geometry. Their findings are as follows:

> *"The responses of Mundurucu adults and children converged with that of mathematically educated adults and children and revealed an intuitive understanding of essential properties of Euclidean geometry."* (Izard et al., 2011, p. 9786)
>
> *"In our first task, we found that, in the absence of formal education in geometry, Mundurucu children and adults are able to reason about ideal concepts in accordance with the predictions of Euclidean geometry."*

In short, this research seems to tell us that: "…at all ages, children and adults can use *distance* relationships" and that "adults in both cultures also located a target by analysing two other fundamental properties of Euclidean geometry: *angle* (the information that distinguishes corners of a triangle that differ in size) and *sense* (the information that distinguishes a form from its mirror image)" (Ibid).

The above indicates that distance, angle and sense inform observation and perception of the natural and built environments, providing identification, description, and comparison of the world around us. This is in line with foundational guidelines from curricula, including the South African schools' curriculum, which requires that:

- *Learners describe the position of objects, themselves and others using the appropriate vocabulary.*
- *Learners follow and give directions.*

- *Learners explore properties of 3-d objects and 2-d shapes by sorting, classifying, describing and naming them.*
- *Learners draw shapes and build with objects.*
- *Learners recognise and describe shapes and objects in their environment that resemble mathematical objects and shapes.*

(Curriculum and Assessment Policy Statement, Grades R – 3, 2011, p. 10)

These geometric elements, or rather the *properties* that give them existence, or that they give existence to, form the basis of our engagement with geometry in school mathematics. Without the geometric elements, based on these properties, there would be nothing to discuss, argue, or ponder.

After a brief look into the geometry sections of several national curricula such as Kenya, Namibia, Indonesia, etc., it is clear that there is mention of 2-D shapes and 3-D objects as objects for investigation, or at least for consideration. While some of these curricula mention position and direction, they seldom make it explicit that these include references to 0-D and 1-D. Therefore, as an initial consideration of the concept of dimension, we need to understand what entails and that the 2-D and 3-D specifications exist within a context of all dimensions, which must include 0-D up to 3-D and beyond.

Being able to classify instances within the natural and built environment, whether these exist, or are perceived as 0-; 1-; 2-; and 3-D, sets the tone for classifying all those geometric elements that follow – especially as per school mathematics curricula. Learners use reasoning to classify any of these instances of dimension.

When engaging with 3-D objects, learners should be encouraged to be able use reasoning, applying appropriate criteria, to describe, classify and then name these geometric objects. An inductive approach could be used to allow learners to explore (Singh, 2017) 3-D objects, through comparison and sorting, utilizing one criterion at a time to group the said objects. The intent of such an activity would be to finally realize polyhedrons, and how they may differ from cylinders, cones, hemispheres and spheres. Later, polyhedrons may be sorted according to the number of faces that they have, without considering whether these are prisms or pyramids at that time. Polyhedrons can be sorted into prisms and pyramids as a subsequent activity. This should develop a move away from prototypical regard for some geometric objects. A cube, for instance, is also a hexahedron, and is also cuboid, while at the same time is a square prism as well. Similarly, a pentagon-based (pentagonal) pyramid is also a hexahedron, like the cube, having six faces.

Two-dimensional (2-D) shapes should be treated in the same way, allowing learners to isolate polygons from "the rest." Realizing polygons to be closed, 2-dimensional shapes with only straight sides (line segments) as a point of departure will allow them to classify non-typical shapes as pentagons, hexagons, and so on, thus moving away from the typical regular polygons which represent these shapes on charts in primary school classrooms.

With this focus on properties as a basis for classification, it is hoped that learners will be able to reason through these properties and be able to grasp ideas that require an understanding in terms of inclusivity when regarding geometric object and shapes. Learners will in all likelihood not respond with discomfort and confusion when they are told that all squares are rectangles.

Measurement plays an ongoing and significant role in the identification and definition of geometric objects (Smith & Barrett, 2017), thus constituting the next "big idea" in Geometry as taught in schools. Measurement therefore provides the means for *properties* to be described (Herbst, Gonzalez, & Macke, 2005) at various degrees of accuracy by measurement, by indirect or direct comparison.

It is imperative that the materials that are utilised for the teaching and learning of measurement show the interconnectedness between properties (Smith & Barrett, 2017) to be measured, and the units of measure to be used to qualify or compare those properties across various contexts.

The word "geometry" itself translates into "earth-measure," or measurement of the earth (Shmoop Editorial Team, 2008). Considering that properties, as a geometric gaze, are observed within the natural and built environment which constitute elements of the earth as we know it, we can see that measurement is an integral part of Geometry. In school mathematics, measurement can be defined as *'a number that indicates a comparison between the attribute of an object being measured and the same attribute of a given unit of measure.'* (Van de Walle et al., 2015). Measurement thus serves as a critical link between Geometry and Number, with Geometry dependent on Number in terms of quantification, and Number often dependent on Geometry for context.

Once prototypes of geometric elements have been established, the ability to mentally *visualize* the same, similar, or different objects, and resulting spatial relationships can start to develop. This ability (to visualize) also continues to play an important role in the development of spatial reasoning. Once geometric objects are engaged with, and rationalized, from the point of their properties (importantly), a realization of sameness and difference can be attained. In geometry, the importance of *invariance* becomes evident. Invariance is described as a property of mathematical objects which remain unaltered after certain operations such as certain transformations are applied to such mathematical objects (Zisserman et al.,1995).

Johnston-Wilder and Mason (2005) suggest that *invariance* is a major theme in Geometry: *"In order to see, hear or feel, people need to experience both change and something to contrast with that change, namely, invariance. Consequently, invariance in the midst of change is a central theme in mathematics, and particularly in geometry."* This is true for primary school geometry, where learners will be able to recognise and explain which shapes and objects are the same, and which are different. If there were only one example of any shape, for instance, there would be nothing to compare it with. However, if the same shape was transformed or visualized as transformed, learners would need to be able to distinguish if it was still the same shape or not, as is was, and as it is. Invariance, with regard to the entirety for each of the shapes and objects, therefore, can only be a conception in the presence of transformations, where any transformation is a rigid motion.

Geometric elements at primary school level, if subjected to rigid *transformations*, as mentioned, retain their properties. Of course, these geometric elements can lose some or all of their properties when subjected to transformations. For the purposes of school geometry, transformations generally maintain the properties of geometric elements.

Van De Walle et al. (2013, p. 419) describe transformations as *"changes in position or size of a shape: movements that do not change the size or shape of the objects transformed…"* and goes on to define these transformations as 'rigid motions' (translation, rotation and reflection).

Quite often, in earlier grades, transformations are treated as arbitrary activities where learner may need to transform a shape (through rigid motions) or identify the transformation that it underwent.

Recognizing and describing changes in location or orientation in terms of: points, lines, shapes or objects are basic activities learners are exposed to in the early grades.

Note that transformations with points are included here. There is a suggestion that if points can be "successfully" transformed, it may be that learners will recognize points (or lines) within shapes and transform any shape according to its points – instead of considering only the shape and trying to transform it as such.

Issues of symmetry need to be included as well, and how symmetry may be evident when performing transformations (Fife, James, & Bauer, 2019). Transformations can result in tessellations and provide excellent application activities for exploring shapes and objects.

Hence, once the three content "big ideas" had been established, underpinned by core knowledge, and held together by conceptions of invariance and the ability to visualize and reason, the way was paved to develop sets of content and practice standards, which are deemed as necessary to develop the requisite levels of spatial reasoning within learners.

Practice Standards for Geometry and Measurement

While some of the standards listed here may be specific to Geometry, most are common across all content areas, and are included in the Mathematical Thinking standards.

- Knowledge of visualizing
- Knowledge of reasoning and justification
- Knowledge of generalizing geometric ideas
- Knowledge of classifying and defining
- Knowledge of investigating invariants
- Analysing and interpreting a figure
- Knowledge of technology
- Mixing deduction with experimentation
- Knowledge of Mathematical Problem Solving
- Dispositions in terms of learning and teaching mathematics

Content Standards

The knowledge standards are derived from three identified "Big ideas" in teaching Geometry and Measurement, underpinned by the core knowledges that are intrinsic to human beings, namely,

- Foundational/Core Knowledges
- Knowledge of Geometry Properties
- Knowledge of Transformations
- Knowledge of Measurement

Foundational Standards

Foundational Standards relate to an awareness of core knowledge which learners possess prior to the commencement of former schooling.

The identified sub-standards are:

- 5.1: An awareness of position (location), distance (length), direction (angle), and “sameness” (invariance)
- 5.2: Ability to describe position relative to other positions or markers
- 5.3 Estimation and comparison of distances and lengths (magnitude)
- 5.4 Ability to indicate direction or describe an angle in terms of directions
- 5.5 Awareness of sameness and difference and similarities
- 5.6 Recognition of invariance after transformations

Content Standards for Geometrical Properties

In Geometry, we may assume that the properties of geometric elements make these elements to be what they are. Similarly, Figura (2007, p. 73) maintains “Geometric properties are those that can be derived from the geometry of a solid body or particle.” In other words, if any element should lose, or change any of its properties, it will no longer be what it was. It will not remain invariant; it will be transformed.

This suggests that one of the most important ideas underpinning the teaching and learning of geometry in school mathematics is those *properties* which define geometric elements to be what they are.

Further, properties are most likely to be those attributes which are initially perceived when any geometric element is observed. These involve:

- Understanding dimension;
- Rational classification of 3-dimensional objects according to observed properties;
- Rational classification of 2-dimensional shapes according to observed properties;
- Realisation of inclusivity with regard to objects and shapes; and
- Economy of definitions.

Examples of activities relating to *properties* from the PrimTEd Geometry and Measurement Working Group:

GM 1.6 **- Realising That 3-D Objects Can Consist of Flat and/or Curved Surfaces**

For 3-D Objects:
Hand out a variety of geometric objects to the students. These should include spheres, hemispheres, cylinders, cones, prisms (cubes, rectangular, square, triangular, hexagonal, pentagonal, octagonal`), pyramids (square-based, triangle-based, hexagon-based, etc.).

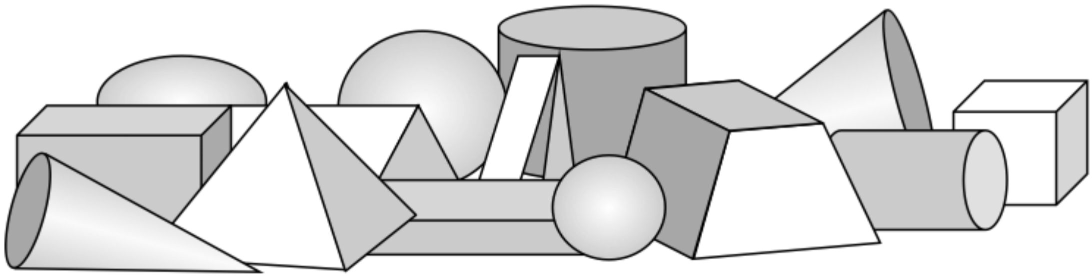


Figure 2: 3-D Objects

Students can be asked to sort these objects into THREE groups. Give reasons why they have grouped them in this way.

Students should be asked to research the Van Hiele Levels and say which level they think that they are operating at in terms of spatial reasoning. It is recommended that students be provided with an accessible text which outlines the Van Hiele Model. In mathematics education, as discussed under the theoretical framework previously, the Van Hiele Model is a theory that describes how students learn geometry.

We have noticed that students tend to sort the objects into "those that look like triangles" or "those that are like circles" or "those that are like blocks." You may point out to them here that they may be operating at Level 0 of the Van Hiele Model!

The three potential groups that the students sort the objects into are:

- Those with only curved surfaces (spheres, etc.)
- Those with only flat surfaces (polyhedrons)
- Those with curved and flat surfaces (cylinders, cones, hemispheres)

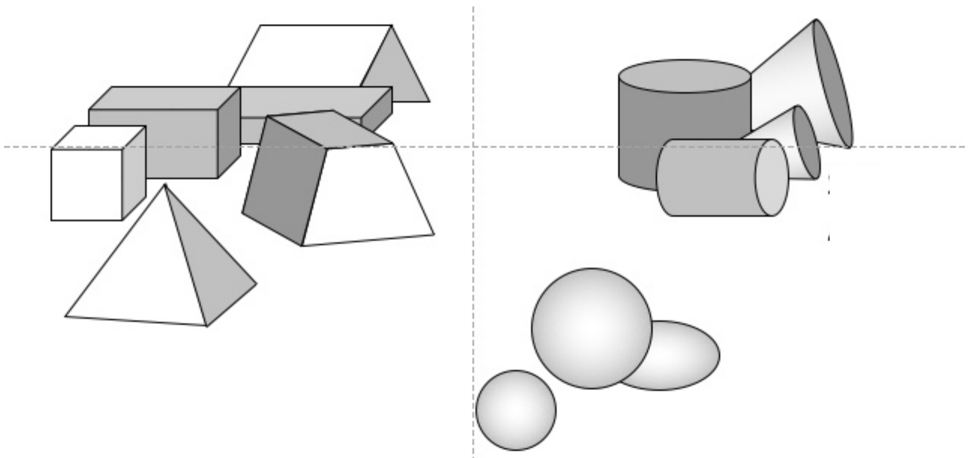


Figure 3: 3-D Objects Grouped Into Three Shapes

Let's take a closer look at those objects which have only flat surfaces.

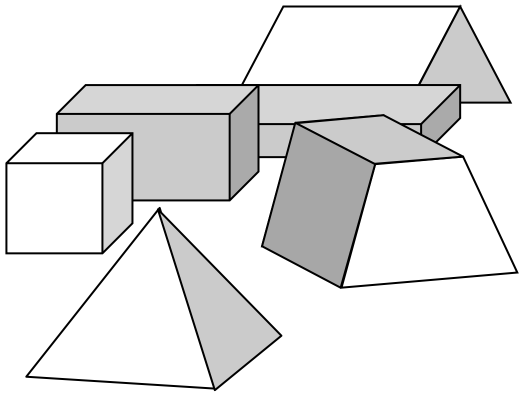


Figure 4: Flat Surfaces 3-D Objects

- These objects are solids, and they can only slide because they have flat faces.
- These are flat-faced solids.
- In Mathematics, flat-faced solids are called *polyhedrons* (or polyhedra).
- "Poly" means *many*, and "hedron" means *faces.*
- Polyhedrons are therefore, 3-dimensional flat-faced solids.

It is very important that the focus remains on the properties, and that all objects have the same property/properties.

For 2-D Shapes:
Hand out a variety of shapes to your students. These should be separate, and printed onto pages so that students can sort, arrange, re-arrange, as they discuss groupings.

To streamline this activity, you can ask learners to remove all shapes that are open, then remove shapes that have curved sides (sides that are not straight).

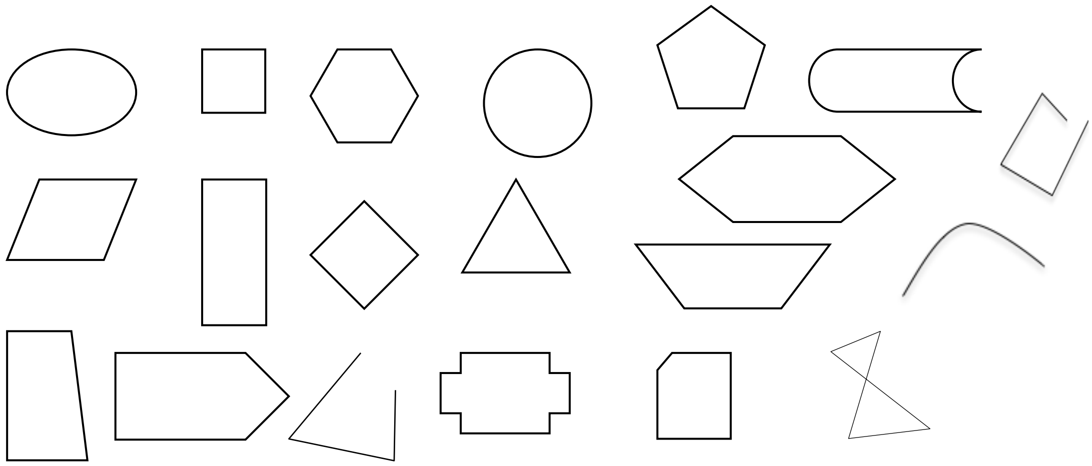


Figure 5: 2-D Shapes

Using the shapes that you've been given, sort them accordingly:
1. Remove all the shapes that are open
2. Now you have only a group of closed shapes
3. Remove all the shapes that have curved or "wavy" sides
4. Now you have a group of closed shapes with only straight sides

This activity will set out to get learners to realise *polygons* as closed, 2-dimensional, straight-sided figures, through a process of sorting.

The facilitator of this activity may wish to allow the learners to sort the provided shapes – with reasons given for every grouping, without providing any guidelines in terms of the properties – allowing learners to "discover" the polygons. Of course, the facilitator will wish to reach a grouping of closed, 2-dimensional, straight-sided figures. While there may be one or two complex polygons among the shapes to be sorted, these should present a challenge in terms of later classification according to the number of corners/sides. The facilitator can explain that complex polygons are conceptions which exist, but will be dealt with during further investigations in mathematics.

Content Standards for Measurement

Emphasizing relations between different applications of measurement (involving length, area, volume, capacity, time, mass, etc.) is critical in developing a robust conception of what measurement is. The identified sub-standards relate to:

- The ability to recognize and isolate the (measurable) attribute of the object being measured
- The ability to select a unit that correlates with the attribute being measured
- Recognizing the cardinality of the units employed
- Realizing that a measure is constituted through iterating the selected unit
- The ability to employ estimation as a means to demonstrate an understanding of units and the measurement process
- Understanding the relation between Number and Measurement

Examples of activities relating to *measurement* from the PrimTEd Geometry and Measurement Working Group:

GM 2.2.1 - Select a Unit That Correlates (Dimensionally) With the Attribute Being Measured

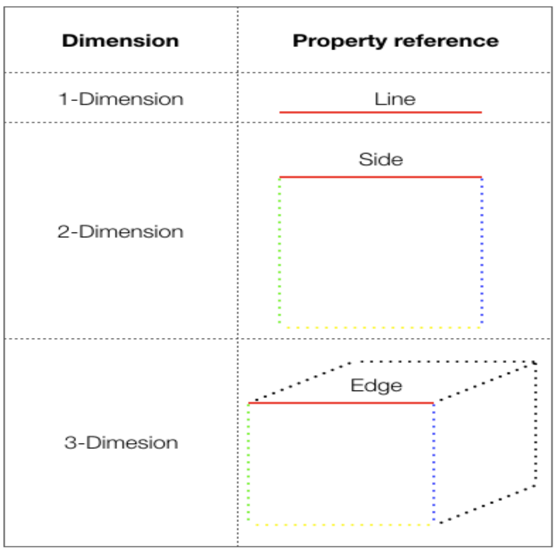


Figure 6: Unit and Their Attribute Measurement

Choose the type of unit you would use to measure (choose from A, B, or C):

Figure 7: Measuring Units

- The size of the floor in your kitchen
- The distance to the office
- The amount of space inside a cupboard

GM 2.3.1 - Recognizing the Cardinality of the Units Employed

GM 2.4.1 - Realizing That a Measure Is Constituted Through Iterating the Selected Unit

- What is the length of the umbrella below?
  Using the counting stick as a unit, measure the approximate length of the umbrella.
  Standardizing the Unit.

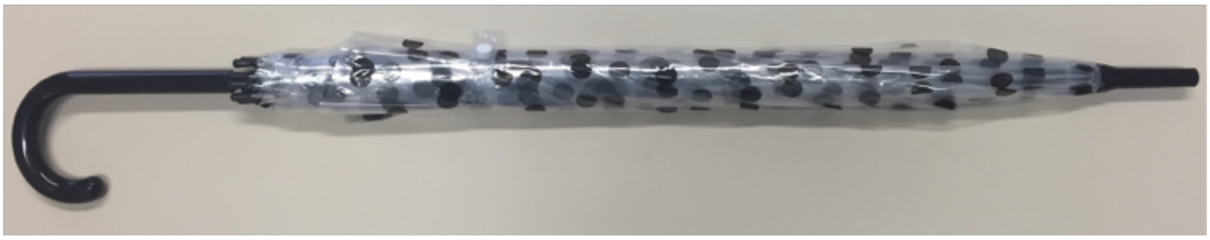


Figure 8: Umbrella

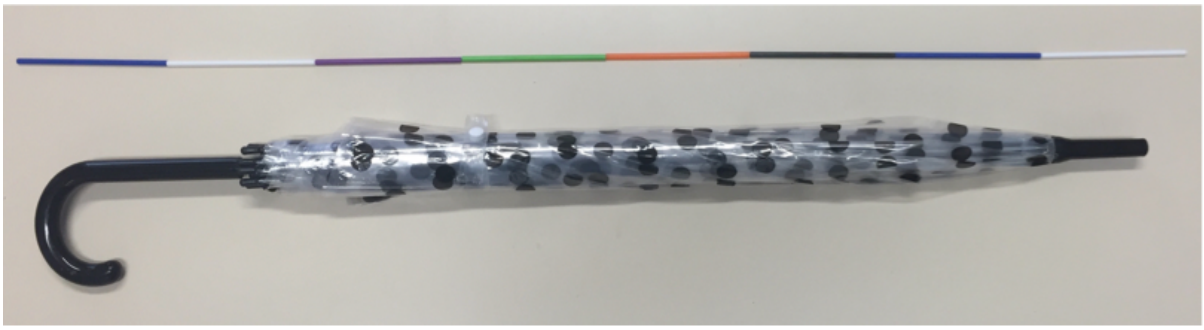


Figure 9: Umbrella and Counting Sticks

The umbrella is approximately 8 counting sticks long.
Using a stack of counting blocks as a unit, measure the approximate length of the umbrella.

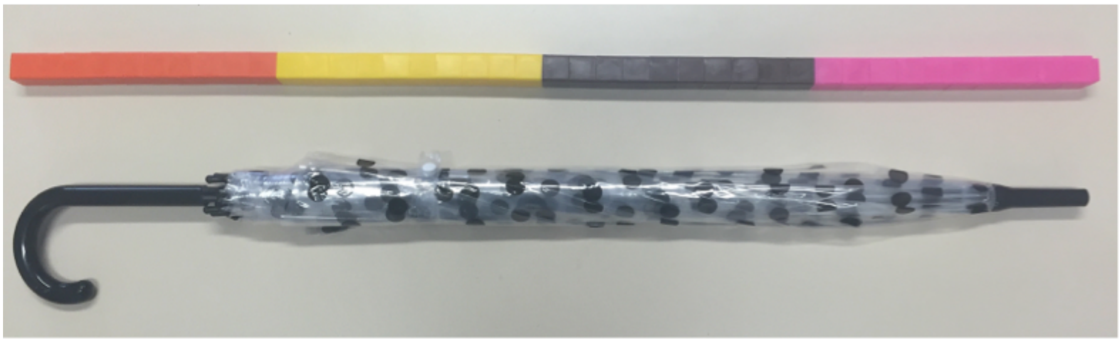


Figure 10: Umbrella and Counting Blocks

- The umbrella is approximately 4 counting block stacks long, which answer is correct?
  8 counting sticks?
  4 counting block stacks?
- How do the sizes of the counting sticks and counting block stacks compare to each other?

Content Standards for Transformations

Knowledge of transformations in Geometry develop the ability to manipulate, visualize, recognize, identify invariance and variance among geometric elements in a variety of orientations and from different perspectives (PrimTed, 2019). The developed sub-standards centre on:

- Understanding and representing translations, reflections, rotations, and dilations of objects in the plane
- Drawing and constructing representations of tessellations of two-dimensional geometric shapes or three-dimensional objects using transformations and a variety of tools
- Comparing geometric patterns (tessellations) that share common characteristics (e.g. form, line, angle, vertex arrangement, space)
- Demonstrating how (elements and principles) can be used to solve specific spatial visual problems
- Planning and producing works of art applying mathematical techniques, and processes with skill, confidence, and sensitivity

Some examples of activities relating to *transformations* from the PrimTEd Geometry and Measurement Working Group are shared below:

GM 3.2.1 - Identify Similarities and Differences Between Geometric Patterns (Tessellations) That Share Common Characteristics (e.g. Form, Line, Symmetry, Angle, and Vertex Arrangement, Space)

GM 3.2.2 - Construct Representations of Transformations of 2-D Geometric Shapes and/or 3-D Objects Using Tessellations

What do you think the formations **4.4.4.4** and **8.8.4** refer to regarding tessellations?

- A particular tessellation (arrangement of shapes) is named by observing a **vertex point** and ascertaining **how many** polygons **touch** the vertex point.
- Conventions are named based on the **type of polygons** that touch the vertex point.
- The convention number represents the **number of sides** of each polygon.

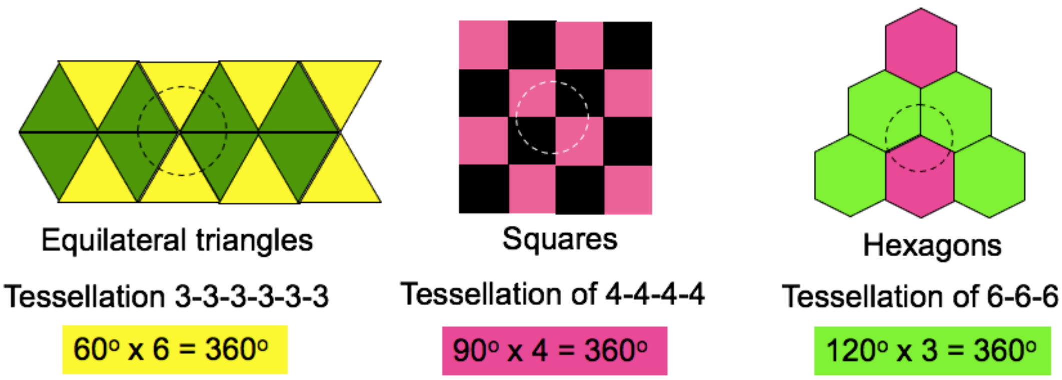


Figure 11: Tessellations of Geometric Shapes

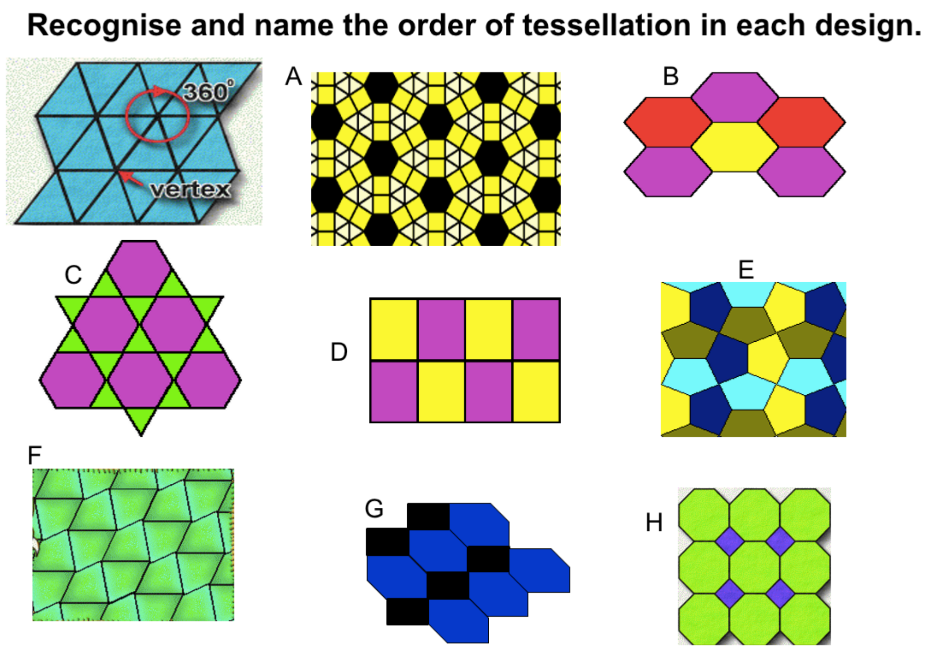


Figure 12: Tessellations Used in Designs

GM 3.3.1 - Use Properties of Objects and Shapes in Relation to the Principles of Transformations to Solve Spatial Problems

1. How many of the small squares (as seen in the figure) would make up the area of the entire tangram?

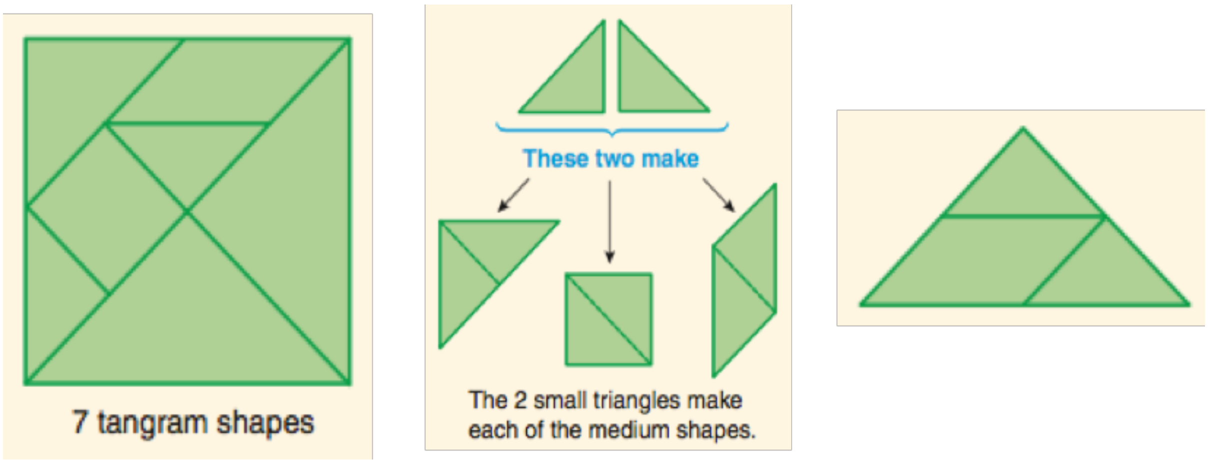


Figure 13: Transformations in Tangram Shape

2. Complete the table below. Discuss your findings in your groups. Write down your main findings.

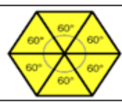
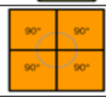
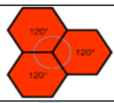
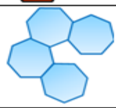
Regular shape	Name	No of angles	Size of one angle	Sketch the arrangement of angles around a point	Do angles round a vertex-point add up to 360°	Is the angle size a factor of 360°	Does the shape tessellate? YES/NO
							
							
							
							
							

Figure 14: Group Activity

3. Study the shaped below. Copy each shape and draw the image of each under a reflection in line *a*

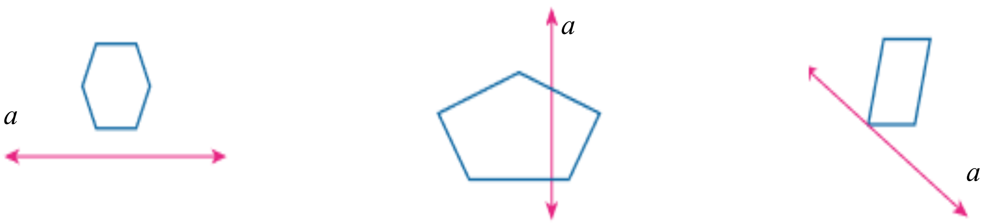


Figure 15: Shapes

Transformation, as reflected by the above activities, is an explicit and deliberate inclusion, seeking to develop a specific concept level as part of the process of developing student-teachers' understanding of geometry. Transformation as a mathematical concept incorporates the development of spatial reasoning as stated previously, which is a cross-cutting concept. Teacher-educators should be mindful when mediating the content with to avoid a blinkered conception of the concept of transformation and its utility in understanding the field of mathematics.

Conclusion and Recommendations

The three content "big ideas" may deviate from other research-based theories. The main reason for the "big ideas," as proposed by this study, to be content-based is that cognizance was taken of the influence of national school curricula as an organizing and guiding factor for pre- and in-service teachers. It was felt that pre-service teachers should be able to recognize what they needed to teach in the schools, and see this reflected in the curriculum, as well as in the salient content that they had covered during their lectures.

The "big ideas" could well have been *invariance* and *visualization* as suggested by members of the working group at initial meetings, which could have privileged the desired *habits of mind* which are so necessary to develop the required dispositions with regard to Geometry.

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