Exploring High School Learners' Proficiency in Euclidean Geometry

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Abstract
This paper reports on a qualitative study that probed high school learners' proficiency in Euclidean geometry in South Africa. Euclidean geometry lessons were conducted in a collaborative learning classroom, and participants' competence was assessed using Kilpatrick's five strands of mathematical proficiency as benchmark. Kilpatrick's five strands of developing mathematical proficiency was the theoretical framework and it was also used to inductively analyse participants' oral and written responses from the collected data gleaned from participants' oral and/or written responses to activities, investigation tasks, Mathematical proficiency test and classroom observations. This study established that majority of participants had challenges in all the five strands. The few participants who demonstrated competence and proficiency in Euclidean geometry provided substantial evidence of mastery of all the five strands attesting to the assertion of inter-dependence of Kilpatrick's five strands of mathematical proficiency. The researchers concluded that students lacked proficiency in Euclidean geometry, therefore, recommended that appropriate strategies must be implemented during mathematics lessons to assist students to develop all the five strands of proficiency in Euclidean geometry, as this will assist them to overcome their mathematical learning difficulties and under-achievement.

Keywords: Mathematical Proficiency, Euclidean Geometry, Problem-Solving

Introduction

Teachers, in attempting to ensure students attain efficiency in learning mathematics, has resulted in several vital reconceptualization shifts in classroom practices. The most recent evolution being the development of "mathematical power", that is, cultivating students' competence in reasoning, problem-solving, connecting mathematical ideas and communicating mathematically to others, for the development of "mathematical proficiency" (NRC, 2001; Schoenfeld, 2007; Brijlall & Ally, 2020; Corrêa & Haslam, 2021; Brijlall & Ivasen, 2022). This does not imply that developing students' mathematical knowledge, basic skills, numeracy, and computational skills which preceded the concept of developing students' mathematical power were superfluous. Rather, they were considered insufficient in addressing students' challenges in learning mathematics. Thus, the focus migrated to conceptual understanding and application of mathematical knowledge (NRC, 2001; Corrêa & Haslam, 2021). According to Schoenfeld, (2007, p.64) "knowing mathematics, in the sense of being able to produce facts and definitions, and execute procedures on command, is not enough; rather, being able to use it in the appropriate circumstances is an essential component of proficiency". Schoenfeld, (2007, p. 60-68) classified attaining mathematical proficiency, in four dimensions, namely, developing the knowledge base; developing relevant strategies; developing metacognition and applying what one knows effectively, as well as developing students' beliefs and dispositions.

The concept of developing mathematical proficiency as an appropriate mathematics instructional approach was born in 2001, by Kilpatrick, Swafford, & Findell. It encompasses five strands, namely, conceptual understanding, procedural fluency, strategic competence, mathematical reasoning, and productive dispositions. According to Kilpatrick et al., (2001) these strands are not independent, but rather, intertwined, thus, mathematical proficiency cannot be achieved by only focusing on some aspects of the five strands. This implies that developing mastery in all the five strands is paramount (Corrêa & Haslam, 2021). NRC, (2001) embraced the concept of mathematical proficiency as the modus operandi for effective mathematics learning – competence, knowledge, and facility. NRC, (2001) further asserted that developing students' mathematical proficiency in the early grades can assist them to cope with higher-order concepts in later grades, thus, should be nurtured. This highlights the need for students to be mathematically proficient as it will assist them in higher education, workplace, and daily lives (NRC, 2001; Brijlall & Ivasen, 2022).

Attaining mathematical proficiency is not instantaneous, but rather, it develops over time (NRC, 2001). Researchers have noted with concern that "how" mathematical proficiency can be achieved in different contexts is limited in literature, hence, mathematics education researchers have recommended that there is a dire need to teach, differently, mathematics in general and Euclidean geometry, in particular, to assist students to be mathematically proficient (Brijlall & Ally, 2020; Abakah & Brijlall, 2022). This advocates "the reinvigoration of the teaching of mathematics in its entirety – classroom learning practices, content, teaching and assessments" (DoBE, 2018, p.3). South Africa mathematics teachers, hence, are urged to 'un-teach' inefficient instructional approaches which might be the root cause of students' underperformance in mathematics and 're-teach' integral mathematical concepts such as Euclidean geometry through problem-solving and developing students' mathematical proficiency in those concepts. In effect, students will be dissuaded from using ineffective approaches of learning and concentrate on efficient ways of learning mathematics. This will assist them to become good learners, good problem-solvers, and good thinkers

(Fahim & Eslamdoost, 2014) which can give them the opportunity to engage in more comprehensive practices (Swartz & Reagan, 1998).

Milgram, (2010) cited precision, stages, and problem solving as procedures that are essential components of mathematical proficiency; this author established that problem-solving is indispensable in cultivating students' mathematical proficiency, thus, must be taught. Posamentier, Smith and Stepelman (2010) aver that teachers who aim to teach through problem-solving must ensure that students' prior knowledge is adequate, that relevant mathematical techniques are developed, sufficiently practised, and applied abstractly, based on deduction and logic. Otherwise, students will be incapacitated and unable to cope during problem-solving and learning higher-grade mathematics (Brijlall, 2015; Abakah & Brijlall, 2022), hence, a curriculum, which teaches students thinking and problem-solving skills is sacrosanct (Abakah & Brijlall, 2022). This will require students to engage in explicit organized thinking about mathematical concepts and enable them to facilitate reflections on problem-solving, although the systematicity and profundity of teaching thinking is a challenge (Fahim & Eslamdoost, 2014).

According to NCTM, (2000), problem-solving stimulates mathematical reasoning and understanding for learning new mathematical knowledge; this reiterates the need to teach mathematics through problem-solving. The concepts of teaching and learning mathematics via problem-solving and developing students' mathematical proficiency have empirically been established as indispensable mathematical practices (DoBE, 2018). Problem-solving is gaining grounds as an efficacious instructional approach (Abakah & Brijlall, 2022; Syarifuddin & Atweh, 2022; Ofori-Kusi, 2017; Phuntsho & Dema, 2019; Mwelese & Wanjala, 2014). Developing students' mathematical proficiency has globally been established as relevant instructional aims and/or objectives in any mathematical context (Maharaj, Brijlall & Narain, 2015; DoBE, 2018; Corrêa & Haslam, 2021). For instance, the Mathematics Teaching and Learning Framework (MTLF) in South Africa was developed and underpinned by the two dimensions (see Figure 1). Specifically, developing students' proficiency in mathematics have received global acclamation as an effective medium for addressing associated teaching and learning difficulties which have adversely contributed to students' challenges and under-achievements in mathematics, although similar attention is limited in literature on Euclidean geometry (Brijlall & Abakah, 2022). The researchers, thus, aim to explore students' proficiency in Euclidean geometry as they learn and solve problems.

In realising this aim, the researchers adopted the five strands of developing mathematical proficiency by Kilpatrick, et al., (2001). This was employed as an analytic tool to measure participants' achievement in Euclidean geometry-knowledge, skills, abilities, and beliefs. To this end, the researchers focused on the use of the five dimensions as the medium for determining participants' proficiency in Euclidean geometry, thus, the following critical research question was formulated: *Which of the five strands of Kilpatrick's measure of mathematical proficiency are demonstrated by the participants as they learn and solve Euclidean geometry problems?*

Literature Review

The teaching and learning difficulties associated with mathematics generally and Euclidean geometry have challenged South African teachers and learners for decades. This is ubiquitous in mathematics classrooms. A plethora of research studies have been conducted by mathematics education researchers, using a variety of strategies and research designs.

However, the teaching and learning difficulties in relation to Euclidean geometry are still prevalent, to the concern of all and sundry (DoBE, 2018). These prompted the Department of Basic Education in South Africa to realise the need to approach this conundrum differently. To this end, the Mathematics Teaching and Learning Framework (MTLF), see Figure 1, was developed as a possible solution after expert advice in collaboration with literature. This approach aimed at developing students' problem-solving and mathematical proficiency, which have been identified as measures for addressing students' teaching and learning difficulties and under-achievement in mathematics.

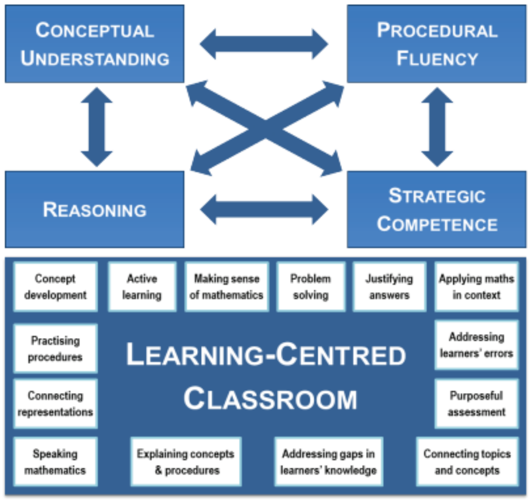

Figure 1: Mathematics teaching and learning framework for South Africa (DoBE, 2018, p.9)

This framework was introduced to South African mathematics teachers to enable them employ appropriate, relevant, and efficient teaching and learning strategies and approaches in mathematics classrooms. It was a follow-up to the Curriculum and Assessment Policy Statement (CAPs) document. The foci of this framework are to guide teachers to teach mathematics to learners effectively; to enable learners develop conceptual understanding, procedural fluency, strategic competence; to develop learners' ability to formulate, present, and decide on appropriate strategies to solve mathematical problems, mathematical reasoning skills, as well as to promote a learner-centred classroom (DoBE, 2018).

Theoretical Framework

Developing Mathematical proficiency by Kilpatrick, et al., (2001), was adopted as the theoretical framework for this study. This theory consist of five intertwined strands of proficiency, each, listed and defined by Kilpatrick et al., (2001, p. 116) as follows: (1) Conceptual Understanding - comprehension of mathematical concepts, operations, and relations; (2) Procedural Fluency - skills in carrying out procedures flexibly, accurately, efficiently, and appropriately; (3) Strategic Competence - ability to formulate, represent, and solve mathematical problems; (4) Adaptive Reasoning - capacity for logical thought, reflection, explanation, and justification; (5) Productive Disposition - habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

In the context of Euclidean geometry, each of the five strands, with reference to Figure 2, can be demonstrated using the following exemplar, which required students to prove that: $XY^2 = DY.YC$; $\hat{A}_1 = \hat{B}_1$; $\frac{r^2}{R^2} = \frac{DY}{CY}$ if $\Delta CAY /// \Delta YBX$. These problems displayed in Figure 3, demand - recalling, applying, and connecting basic geometry knowledge, relevant circle geometry theorems and/or converses of theorems, appropriate geometry concepts, symbols, geometry language, as well as skills. These are all students' prior knowledge of geometry which are highly essential for solving these higher-order geometry problems and eradicating common errors. According to Kilpatrick et al., (2001), students who can demonstrate those competences have attained conceptual understanding. In addition, learners who can apply relevant skills, knowledge and geometric procedures must also know when and how to use them. This is what Kilpatrick et al., (2001) referred to as "procedural fluency". Students' proficiency in Euclidean geometry also requires them to demonstrate strategic competence - formulating geometric problems, representing geometric problems mathematically, using geometric language, notations, and strategies correctly in solving geometry problems. In the context of Euclidean geometry, this require extensive practice, to be able to attain such level of mastery.

In collaboration with the above three dimensions of proficiency, crucial to attaining proficiency in Euclidean geometry is students' ability in adaptive reasoning, which is advanced geometry thinking by reflecting, explaining, justifying, and authenticating the reasonability of proposed solutions to higher-order, non-routine geometry problems (Kilpatrick et al., 2001). This is because Euclidean geometry by its nature is a thinking-laden mathematical concept (Abakah & Brijlall, 2022), thus, students must cultivate logical, creative, critical, and reflective thinking to be proficient in Euclidean geometry. The last competence that needs to be demonstrated by students, according to Kilpatrick et al., (2001) is productive disposition. As seen in Figure 3, this is a higher-order Euclidean geometry problem-solving task which requires persistence, belief, confidence, self-efficacy, and resilience as students search for meaningful solutions to given problems. The process is aided by students acquiring mathematical sensibility and realising its usefulness and worthwhileness. In other words, students must know the value and experience the need for mathematics (Kilpatrick et al., 2001).

Research Design

This study implemented a qualitative case study research design and data was mainly generated through classroom observations and proficiency test conducted. The researchers considered a qualitative case study research design to be appropriate for this study since participants needed to be intensively observed in their natural classroom setting. This design provided the researchers with detailed accounts and explanations of activities ad occurrences at the research field as participants were continuously observed over the period of the research.

Participants

The researchers investigated 32 participants' competence and proficiency in Euclidean geometry as they learn and solve problems; they were from the same class (11B) and were taught mathematics by the same teacher, at the same research field. Participants were taught Euclidean geometry in a collaborative classroom setting. During the lessons, activities, investigation tasks and classroom observations were conducted. Thereafter, a proficiency test was administered. Gender, ethnic, social and race criteria were not employed when identifying participants, therefore, all learners who willingly agreed to participate in this study were allowed to do so.

Ethical Considerations

Before the commencement of this study ethical procedures were adhered to – informed consent, confidentiality, and voluntary participation. On the aspect of informed consent, permission was obtained in writing from the Provincial Department of Education, the SGB of the research field school, participants and their parents or guardians. Anonymity was adhered to as participants' identifications were not revealed. To ensure voluntary participation, only learners who willingly availed themselves and had signed forms of consent, were taken as participants for this study.

Qualitative Data Analysis

This was a qualitative study. Creswell (2012) aver that in a qualitative study, these four steps are sacrosanct: preparing data, analysing data, reporting results, and interpreting the results. According (Creswell & Creswell, 2018) qualitative research investigates meanings individuals or groups attribute to a social or human problem. These authors further posit that, in a qualitative study- emerging questions, procedures, data collected in the participant's setting, inductive data analysis from emerging themes, which are then interpreted for concise and meaningful inference are paramount.

In this study, after the researchers administered the proficiency test to each participant, they marked each participant's script and content analysis was also conducted on each participant's script. Thereupon, data analysis of each participant's written responses commenced; this was conducted in four phases; the procedure followed in this study is illustrated on the arrow diagram below and each phase is delineated thereof.

Participants written responses →Data reduction →Categorizing, Coding and Tabulation → Developing patterns and themes.

Phase '1- Participants' Written Responses

Participants' written responses, attesting to 'how' and 'why' each of the five strands was attained and/or not attained are presented. In this regard some participants' written responses were taken at random and displayed, content analysis and discussion of participants' work were also carried out.

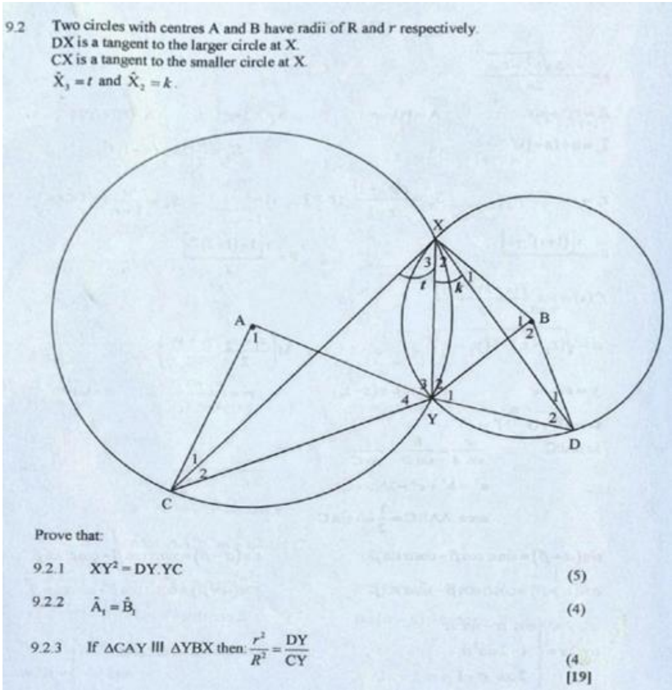

Figure 2: Proficiency test – non-routine task

Figure 2 is a non-routine circle geometry task. This demands applications of the combination of deductions, brainstorming, logical reasoning and advanced mathematical thinking around relevant circle geometry theorems and/or converse of theorems, as well as appropriate geometric properties in order to conjecture appropriate responses to the given questions.

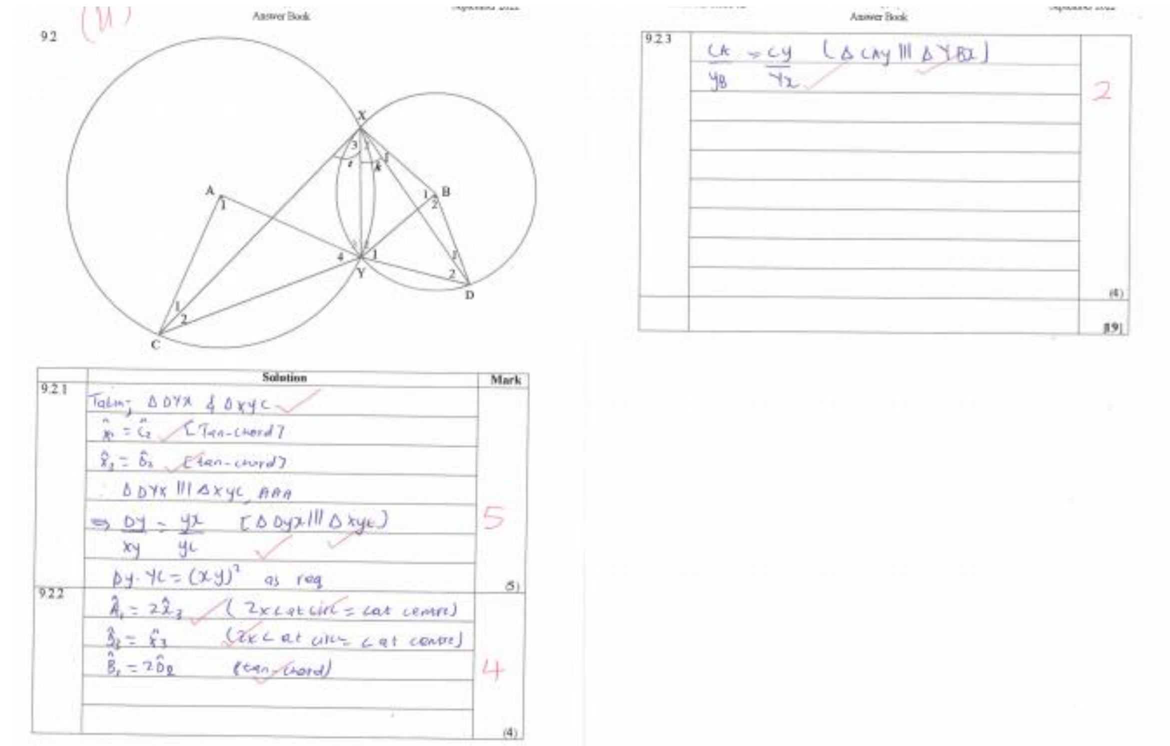

Figure 2.1: A participant's written response

Figure 2.1 informs that this participant was able to interpret the geometric diagram well with reference to the given sub-questions. S/he was able to logically provide appropriate responses to the given sub-questions as s/he brainstormed around the relevant circle geometry theorems and geometric properties. The analysis of this participants written responses displayed in Figure 2.1 is presented on the Mathematical Proficiency indicator rating index form (see Figure 3). Another participant's work, randomly selected, is presented next in Figure 2.2.

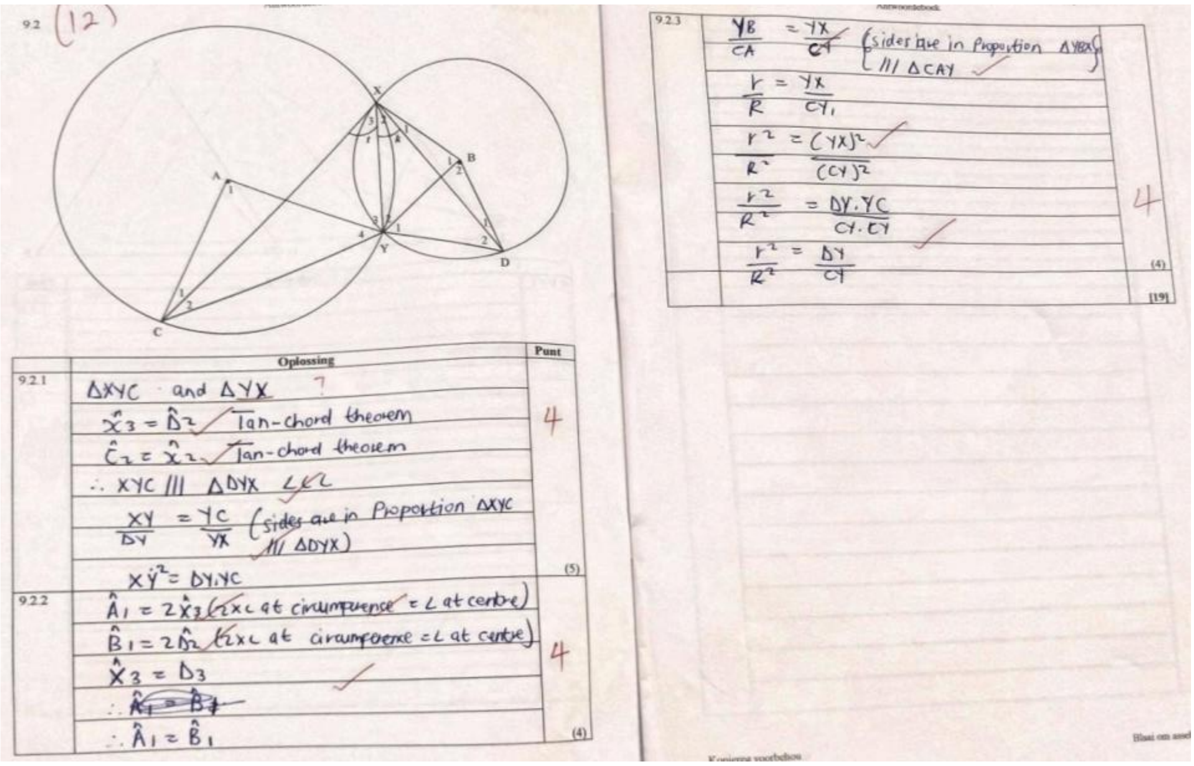

Figure 2.2: A participant's written response

In Figure 2.2 it is observed that this participant was able to interpret the geometric diagram and the sub-questions well. Evidence of logical reasoning and brainstorming around relevant circle geometry theorems and geometric properties can be seen from this participant's work. The analysis of the participant's written responses displayed in scan 2 is like the analysis presented in Figure 3, thus, needed no repetition. The written responses of one other participant, randomly selected, are displayed next in Figure 2.3.

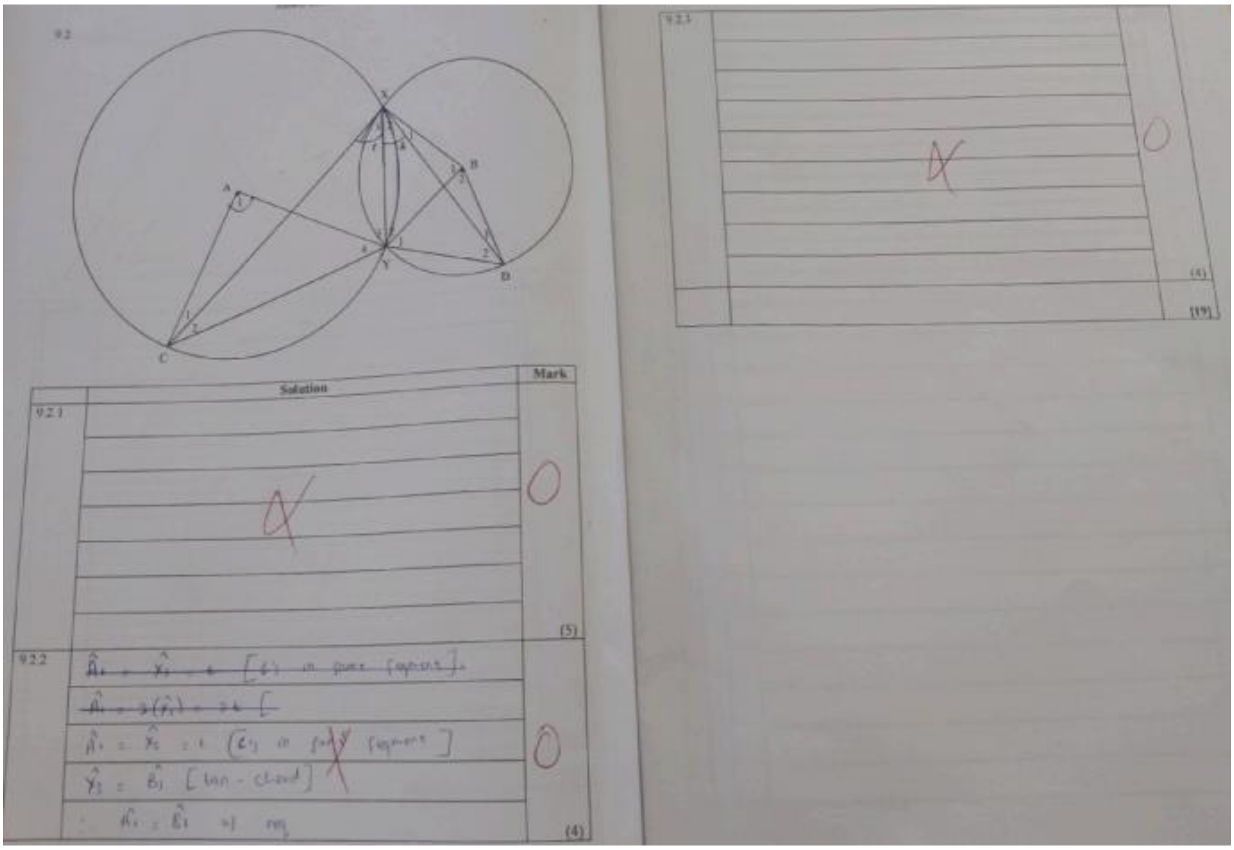

Figure 2.3: A Participant's written response

As displayed on scan 3, it is evident that this participant lacked relevant knowledge of circle geometry concepts and geometric properties. There is no evidence of logical applications of relevant circle geometry theorems, thus, s/he could not make meaningful deductions. This resulted in this participant providing incorrect responses to all the sub-questions - s/he scored zero for all the questions.

Phase 2- Data Reduction

Data reduction is essential for minimising the volume of collected data, so that it can be summarized for easy presentation, interpretation, and analysis (Mezmir, 2020). The collected data was analysed qualitatively by utilising Kilpatrick's five strands of mathematical proficiency by implementing the following five proficiency indicator parameters:

(1) Participants who demonstrated comprehension of mathematical concepts, operations, and relations were rated as having attained category 1- conceptual understanding.
(2) Participants who demonstrated skills in carrying out procedures flexibly, accurately, efficiently, and appropriately were rated as having attained category 2- procedural fluency.
(3) Participants who demonstrated ability to formulate, represent, and solve mathematical problems were rated as having attained category 3- strategic competence.
(4) Participants who demonstrated capacity for logical thought, reflection, explanation, and justification were rated as having attained category 4- adaptive reasoning.
(5) Participants who demonstrated habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy were rated as having attained category 5- productive disposition.

The researchers determined each participant's competence in each of the five parameters by using the Mathematical Proficiency indicator rating index form (see Figure 3) to analysis their written responses. There is need to note that content analysis of a participant's written response was done holistically, hence, a participant could be rated as having attained and/or not attained more than one proficiency parameter as far as evidence of attainment of such proficiency parameter indicators were demonstrated in the participant's work.

After the researchers marked participants' script, data reduction (summary of relevant information) was done (Brijlall & Ivasen, 2022). Thus, the number and percentage of participants who attained each of the five strands from the proficiency test were summarized and presented in a tabular format (see Table 1). This information was obtained from participants' Mathematical Proficiency indicator rating index forms.

Mathematical Proficiency indicator rating index

(1) Conceptual understanding

Indicators: comprehension of mathematical concepts, operations, and relations

~~Attained~~	Not attained

Clarification: All the indicators were demonstrated in the participant's work.

(2) Procedural fluency

Indicators: skills in carrying out procedures flexibly, accurately, efficiently, and appropriately

~~Attained~~	Not attained

Clarification: Indicators were substantially evident in participant's work

(3) Strategic competence

Indicators: ability to formulate, represent, and solve mathematical problems

~~Attained~~	Not attained

Clarification: The participant formulated the given question in his/her own words, represented the question geometrically and provided solutions.

(4) Adaptive reasoning

Indicators: capacity for logical thought, reflection, explanation, and justification

~~Attained~~	Not attained

Clarification: Evidence of geometric thinking was evident in participant's work by applying the above indicators

(5) Productive disposition

Indicators: Habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy

~~Attained~~	Not attained

Clarification: The desire/confidence to find productive, purposeful and justifiable solutions were evident in participant's work.

1 | Page

Figure 3: An example of a completed participant's Mathematical Proficiency indicator rating index form

Phase 3- Categorizing, Coding and Tabulation

The number of participants who displayed the Mathematical proficiency parameter were categorized, coded, and tabulated. This was so that they can easily be compared, from which emerging themes were noted (McMillan & Schumacher, 2014). According to McMillan and Schumacher, (2014, p. 413), "data must be organized to analyse them, either using predetermined categories or developing codes from the data. Predetermined categories are derived from the research problem, an interview guide, literature, and personal or general knowledge". After content analysis of participants written responses was undertaken, data reduction, categorizing and coding of Mathematical proficiency parameters, were followed respectively. Thereafter, tabulation of the number and percentage of participants who operated at each of the five levels of proficiency of Euclidean geometry was summarized and presented (see Table 1).

Kilpatrick's five strands of mathematical proficiency					
Participants Categorization	Strand-1 Conceptual Understanding	Strand-2 Procedural Fluency	Strand-3 Strategic Competence	Strand-4 Mathematical Reasoning	Strand-5 Productive Dispositions
No. of Participants	7	7	7	7	7
Percentages	22%	22%	22%	22%	22%

Table 1: Summary of how participants responded to the proficiency test items based on Kilpatrick's five strands of mathematical proficiency.

As illustrated in Table 1: 7 (22%) of participants demonstrated conceptual understanding, procedural fluency, strategic competence; mathematical reasoning skills and productive dispositions, whereas none of the other 25 (78 %) participants were able to attain proficiency in each of the five strands.

Phase 4- Developing Patterns and Themes

According to McMillan and Schumacher, (2014, p. 412 - 413), "In inductive analysis, the categories and patterns emerge from the data". It can be observed on Table 1 that the number and percentage of participants who demonstrated each category of Mathematical proficiency parameter were presented. The relationship among the number and percentage of participants for the categories is that the same number and percentage attained and/or did not attain all the five categories.

Discussion of Research Findings

In this section, the research findings are presented and elaborated in accordance with the research question drawn up for this study: *Which of the five strands of Kilpatrick's measure of mathematical proficiency are demonstrated by the participants as they learn and solve Euclidean geometry problems?*

This study established that 7 (22%) of participants demonstrated conceptual understanding, procedural fluency, strategic competence, mathematical reasoning, and productive dispositions (see Table 1). Exemplars to testify they are presented in scans 1-2. The completed participants' Mathematical Proficiency indicator rating index form (see Figure 3) inform “how” and “why” the researchers analysed participants’ written responses to assess their proficiency in Euclidean geometry (Kilpatrick et al., 2001; Brijlall & Ivasen, 2022).

It can be seen in Figure 3 that the researchers judged that the participant attained conceptual understanding of Euclidean geometry concepts. The researchers justified this with the clarification that “*All the indicators were demonstrated in the participant’s work”*. This means the participant demonstrated evidence of comprehension of mathematical concepts, operations, and relations. On procedural fluency, the researchers judged that the participant attained this strand of proficiency; the researchers clarified that: *“indicators were substantially evident in participant’s work”.* This implied that this participant could apply relevant skills in carrying out procedures flexibly, accurately, efficiently, and appropriately. The researchers, also, judged that this participant had attained strategic competence; *“the participant formulated the given question in his/her own words, represented the question geometrically and provided solutions”.* In addition, the researchers judged that this participant attained adaptive reasoning; this was supported with the clarification that *“evidence of geometric thinking was evident in participant’s work by applying the indicators-logical thought, reflection, explanation, and justification”.* Lastly, on productive disposition, the researchers judged that the participant attained this proficiency strand; *“the desire/confidence to find productive, purposeful and justifiable solutions were evident in participant’s work”.* According to Kilpatrick et al., (2001, p.116) participants who could show evidence of mastery of conceptual understanding, procedural fluency, strategic competence, mathematical reasoning, and productive dispositions by demonstrating their respective indicators were mathematically proficient.

The narrations above justify “how” and “why” the researchers judged that 7 (22%) of participants demonstrated proficiency in Euclidean geometry. This is because these participants displayed evidence of mastery of the indicators of each of the five stands of Mathematical proficiency. It is illustrated on Scans 1 & 2 that such participants had substantial knowledge of circle geometry theorems, converses of theorems, and properties of geometric shapes and correctly applied appropriate geometric procedures. The contents on scans 1 & 2 also reveal that these participants were able to interpret the geometric diagram well with reference to the given sub-questions. They rightly identified relevant theorems and/or converses and other related geometric concepts; they then logically brainstormed around them and consistently applied chains of deductions and advanced mathematical thinking to conjure appropriate responses to the given questions. These were established as there were ample evidence of advanced, logical, and reflective geometric thinking from these participants’ written responses. Also, the persistence, confidence, and desire to obtain meaningful solutions to the given problems were exhibited; scans 1 and 2 serve as evidence. In these scans, according to Kilpatrick et al. (2001, p.16), these participants demonstrated proficiency in Euclidean geometry as all the five associated strands were mastered and demonstrated in their conjectured solutions.

Notably, the few participants who demonstrated competence and proficiency in Euclidean geometry provided substantial evidence of mastery of all the five strands, confirming the assertion that these five strands of mathematical proficiency are not independent, rather, they are inter-woven and inter-dependent on each other (Kilpatrick et al., 2001; NRC, 2001; Corrêa & Haslam, 2021). This point was established in this study as the 7 (22%) of participants who demonstrated conceptual understanding, also demonstrated competence in the other four strands.

Conclusion

The analyses conducted in this study established that few participants' written responses portrayed that they had factual knowledge of Euclidean geometry concepts, and had understood, applied, analysed, synthesised, and evaluated their conjectured solutions to the given problems. These dimensions are measurement criteria of proficiency in mathematics and extended to Euclidean geometry in this study. Interestingly, the few participants who demonstrated competence and proficiency in Euclidean geometry provided substantial evidence of mastery of all the five strands confirming the assertion that Kilpatrick's five strands of mathematical proficiency are inter-dependent. This study also established that, although, participants were exposed to Euclidean geometry content and problems, majority of them were still unable to attain proficiency as they had challenges in all the five strands. The researchers, thus, deduced from these findings that a participant either demonstrated competence and mastery of all the five strands or none of them. Most participants who did not demonstrate competence in any of the five strands gave the researchers a reason to conclude that students lacked proficiency in Euclidean geometry. It is, therefore, recommended that appropriate strategies must be implemented during mathematics lessons to assist students to develop all the five strands of proficiency in Euclidean geometry as they are inter-dependent. This approach will assist students to overcome their learning difficulties and under-achievement.

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