Mathematical Interpretation of Perspective in a Combine Painting Composed of Multilayer Cuboids

Claire H. Shin, St. Bernard's Academy, United States

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Abstract
Cuboids are one of the most common geometries, and the multilayer cuboid is often used in various sculpture and science. Combine painting is an art that incorporates 3D sculpture into 2D painting and produces various visual effects. A significant component of 2D painting is perspective, and the geometric variation of sculpture affects the visual effect of the art through harmony with the perspective of painting. A geometrical analysis was conducted according to the tilting angle variation of a multilayer cuboid. It was proven that as the angle increases, the overall length and area of the multilayer cuboid in the perpendicular view increase. In the vertical view, the lengths of each cuboid must overlap under a certain condition, and the overlap frequency increases with increasing tilt angles. These geometric analyses can aid in designing multilayer cuboid sculptures that fit the perspective of the 2D art works in the combine painting.

Keywords: Perspective, Multilayer Cuboid, Combine Painting, Angle Variation, Geometrical Analysis

Introduction

Beyond math, science, and art, the cuboid is one of the most used shapes in everyday life. Multilayer cuboid geometry is simple and useful, allowing you to stack cubes of different shapes into structures that can be used to explain mathematical and scientific phenomena, as well as in other fields such as Lego and games. For example, multilayered cuboids are an important engineering element in microelectronic devices (Lau, 1993). Animal skin and plant tissues also have multilayered cubic structures, which play an important role in regulating physiological phenomena or protecting organisms from the outside world (Brocklehurst et al., 2023). Multilayered cuboids are also very interesting geometrically and practically, because by adjusting the size and angles of the cubes, we can create almost any structure we can imagine. Recently, there has been a growing effort to understand these multilayer cuboids mathematically (Sachs et al., 2021; Wang et al., 2020).

Combine painting is a new field that combines 2D painting with three-dimensional sculpture. The three-dimensional effect created by a 3D sculpture object in the painting works of a 2D surface creates various visual effects. However, since the Renaissance, efforts have been made to increase the sense of three-dimensionality in 2D paintings, and the perspective technique is the most representative one (Folland, 2010; Ikegami, 2010). 2D painting uses perspective to represent a three-dimensional surface, and as several point perspectives are used, painting has an excellent visual effect with a variety of three-dimensional effects (Li, 2023). However, once a 2D painting is completed, the perspective is fixed. On the other hand, in terms of sculpture, a 3D objective can be viewed from any perspective. Unlike 2D painting, the perspective changes depending on the viewer's gaze. Therefore, even if one 2D painting has a certain perspective, depending on how the 3D sculpture is incorporated into the 2D painting, the overall three-dimensional effect of the combined painting will vary, which will produce various visual effects. In this respect, interpreting combine painting in terms of perspective is an interesting topic (Soriano-Colchero1 & López-Vílchez, 2019; Conway & Livingstone, 2007). However, unlike the fields of science and engineering, there are not many attempts to mathematically understand the multilayer cuboid used in art.

This study analyzes the geometric phenomena that occur when a group of cuboids align with the perspective of a 2D painting. We analyzed the phenomenon that a one-directional cuboid stack overlaps the length of each cuboid seen perpendicular to the tilt direction according to the tilt angle variation, and the geometrical changes such as the length and surface area of the entire cuboid stack were mathematically analyzed.

Example Combine: Incorporation of a Cuboid Stack Onto a 2D Painting (Example 1)

A multilayer cuboid was designed by adjusting the height of each cuboid of the same width to fit well the area of the ship shown in Figure 1. Twenty-nine cuboids with a width of 0.6 cm, of which length were equal to their heights, successfully fit the area, attached perpendicular to the painting. Looking at the front view, the painting perspective and the vertically lined cuboids are not that awkward in terms of the overall three-dimensional effect. However, when viewed from the top view, the line direction of the cuboids and the perspective of the painting are inconsistent, creating an unharmonious feeling (Figure 1).

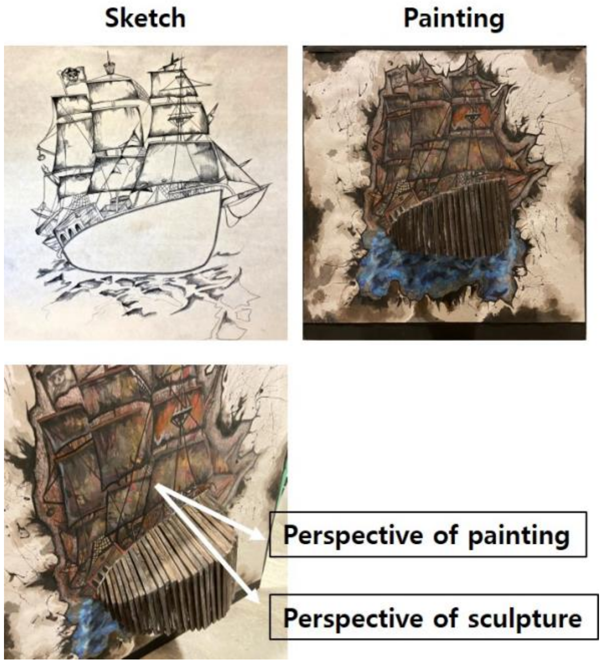

Figure 1: Mismatching of perspective of painting with sculpture
(Case study of Example 1).

Effect of Angle of Cuboid Stack on Variation of Length and Surface Area of the Front-View of the Sculpture (Example 2)

Figure 2 is the geometry of an example multilayer cuboid seen from the top view and front view. Front view is the x-y plane, and top view is the x-z plane. First, looking at the top view, as the multilayer cuboid tilt angle changes, the shape changes, but the overall area does not change.

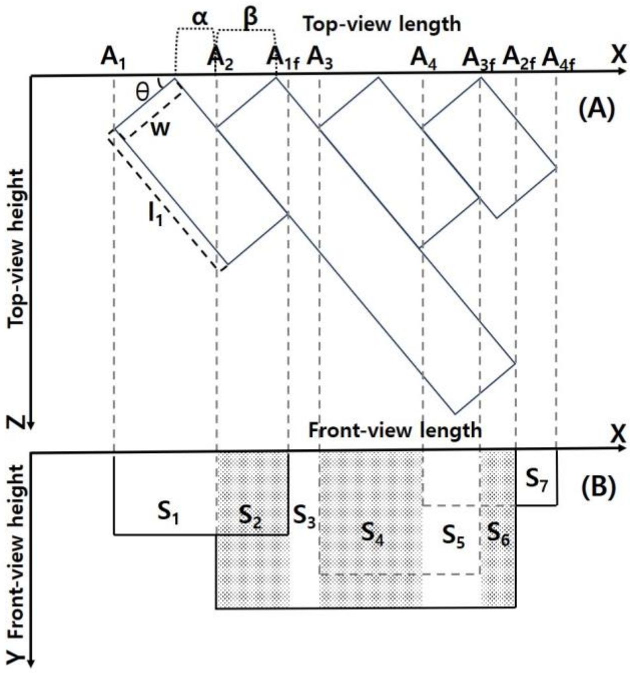

Figure 2: Illustration of lengths of each segmented front-view surface area of
an example cuboid stack (Case study of Example 2).

Top- and front-views both use x-direction. The x-directional movement of a cuboid in the top-view is recognized as movement in the same position in the front-view. Therefore, the x-position and relative distance of each movement of the cuboid in the top view are applied the same as those in the front view. Figure 2 is an example of four cuboids with different lengths and heights forming a multilayer. The starting position point in the x-direction of each cuboid is denoted as A, and the ending position point is denoted as A_f.

As shown in Figure 2. Due to the tilt angle of the cuboid stack, the length (x-direction) and the overlapped surface area in the front view of each cuboid changes, as well as the entire front-view surface area of the multilayer cuboid. In a case, by adjusting the angle of the cuboid stack and aligning it with the painting perspective, the top view could be harmonized with the painting, but surface variation occurs in the front view. Therefore, it is necessary to analyze the effect of the tilt angle on the length and surface area of each cuboid in the front view.

The length of each front-view segmented surface area varies depending on the length and angle of each cuboid that makes up the multilayer (Figure 2). This makes various length combinations of A and A_f, as shown in Table 1.

Cases	The segmented length of the front view surface area
A_i A_{i+1}	$(\alpha+\beta)$
A_{if} A_{i+j}	$[\ \alpha(j-1)+\beta j\ ] - l_i \sin\theta$
A_{i+j} A_{if}	$l_i \sin\theta - [\ \alpha(j-1)+\beta j\ ]$
A_{if} $A_{i+j\ f}$	$l_{i+j} \sin\theta - l_i \sin\theta + (\alpha+\beta)j$
$A_{i+j\ f}$ A_{if}	$l_i \sin\theta - l_{i+j} \sin\theta - (\alpha+\beta)j$

Table 1. Segmented length of the front view with respect to allocation of end points of cuboids.

Mathematical Analyses of Length Variation According to Angle Increase

The Effect of Angle on the Front View Length of the Multilayer Cuboid

The first characteristic feature in the Example 2 case study is that the front view length of each cuboid changes depending on the tilt angle. It is meaningful to check whether there is a general direction for the total front-view length according to the change in tilt angle of the multilayer cuboid. For the non-overlapped multilayer, the total length depends on the last cuboid.

$$\frac{\partial\left(w\cos\theta+(n-1)\frac{w}{\cos\theta}+l\sin\theta\right)}{\partial\theta}=\frac{w\sin\theta}{\cos^2\theta}\left[(n-1)-\ \cos^2\theta\right]+l\cos\theta\ \geq\ l\cos\theta\ \geq 0 \qquad (1)$$

Following Eqn. 1, the total front-view length increases, as the tilt angle of the multilayer cuboid, since the derivative is positive according to angle increase.

In the Example 1 case study, at angles 0, 5, and 15, the 29th cuboid is the final cuboid in the multilayer. At 30 and 45 degrees, overlapping occurred and the 25th cuboid is final in the multilayer. However, even in the case of overlapped cases, Eqn. 1 is equally satisfied, so we can see that as the angle increases, the front-view length increases (Table 2).

Angle (degree)	0	5	15	30	45
Length (cm)	17.4	17.58	18.33	20.75	25.88
Surface area (cm^2)	101.76	105.62	115.19	136.81	171.42

Table 2. Case study of Example 1. Variation of length and surface area of the front-view according to the cuboid stack angle.

Condition for Increasing the Front View Length of the Tilt Angled Multilayer Cuboid Over the Non-tilted Whole Multilayer

Equation 1 shows that the multilayer length increases with angle. In addition, through the Example 1 study, it was confirmed that the overlapping multilayer (25^{th} cuboid in the final) exceeds the length of the non-tilted whole multilayer. Therefore, it is necessary to elucidate the condition at which this happens. For a multilayer with n cuboids, Eqn. 2 is the condition when the cumulated length of the multilayer from 1^{th} to i^{th} cuboid is greater than or equal to that of the non-tilted multilayer from 1^{th} to n^{th} cuboid, the whole multilayer.

$$w\cos\theta + \frac{(i-1)w}{\cos\theta} + l_i \sin\theta - nw = \frac{1}{\cos\theta}[w(\cos\theta)^2 - (nw - l_i\sin\theta)\cos\theta + (i-1)w] \geq 0 \quad (2)$$

Where w, l are width and length of each cuboid, respectively. The condition that Eqn. 2 is satisfied for any value of $\cos\theta$ is $determinant\ (Eqn. 2) \leq 0$ (Eqn. 3).

$$n - 2\sqrt{i-1} \leq \frac{l_i \sin\theta}{w} \leq n + 2\sqrt{i-1} \quad (3)$$

The Effect of Angle on Overlapping of an Angled Multilayer Cuboid Over Another

However, the overlapping found in the example 1 case study could occur not only in the 25th cuboid but also in other cuboids. In other words, there is a need to investigate the impact of angle on the last cuboid seen in the multilayer. The mathematical interpretation of this is as follows. Equation 4 describes the difference of front-view length between the multilayer of last cuboid i^{th} and that of j^{th}, and Equation 5 is the necessary condition for Eqn. 4.

$$f = \left(w\cos\theta + \frac{w}{\cos\theta}(i-1) + l_i\sin\theta\right) - \left(w\cos\theta + \frac{w}{\cos\theta}(j-1) + l_j\sin\theta\right) \geq 0, \quad i < j \quad (4)$$

$$f = (l_i - l_j)\sin\theta - \frac{w}{\cos\theta}(j-i) \geq 0$$

$$\sin 2\theta \geq \frac{2w(j-i)}{l_i - l_j} > 0 \ and \ l_i > l_j \quad (5)$$

Where $l_i\ and\ l_j$ are lengths of i^{th} and j^{th} cuboid, and $i < j$.

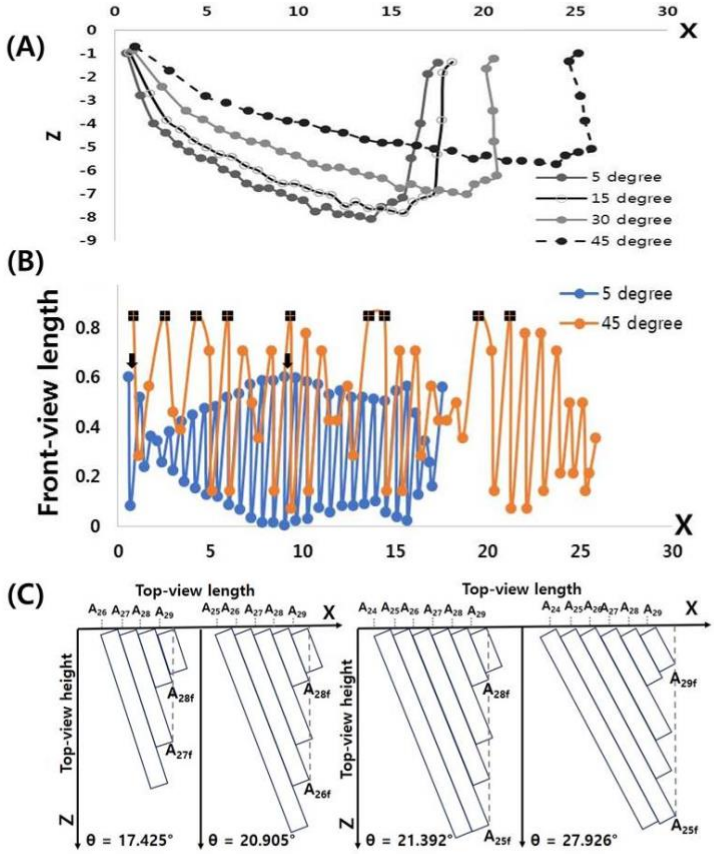

Figure 3: Case study of Example 1. (A) Variation of length and height of top-view of the cuboid stack. (B) Variation of series of segmented length according to the cuboid stack angle.

Rectangle square and arrows: A_i-A_{i+1}. (C) Minimum angles for overlapping cuboid stack.

Equation 5 implies that the front-view length of i[th] in the final cuboid could be larger than that of j[th] cuboid, regardless of less number of cuboids. As seen in Figure 3(A), 25[th] cuboid was the last one in the stack at 30 and 45 degree of angles, satisfying Eqn. 5. Also, Eqn. 5 shows that the potential of overlapping is proportional to $l_i - l_j$ but negatively proportional to the width of the cuboid.

However, when the length and width of all cuboids are determined, angle variation has the greatest influence on the view of the cuboid stack. The effect on overlapping according to angle is shown in Eqn. 6.

$$\frac{\partial f}{\partial \theta} = w(j-i)\sec\theta\tan\theta + \left(l_i - l_j\right)\cos\theta \geq 0, \quad 0 \leq \theta \leq {}^{\pi}/_{2} \tag{6}$$

According to Eqn. 6, it can be seen that as the angle increases, the probability of overlapping increases. Due to overlapping, the number of cases, A_{if} A_{i+jf}, in Table 1 increases, which inevitably increases the frequency of α+β front-view length, the distance between A_i A_{i+1}.

Therefore, as the angle increases, the length series tends to become more complex, which can be confirmed through the Example 1 case study (Figure 3[B]).

Equation 5 can be used to simulate various overlapping conditions. For example, in the Example 1 case study, the minimum angle at which various overlapping cases begin to occur according to the proximity of A_{if} can be calculated using 29 cuboids. The simulation results are shown in Figure 3(C) and Figure 4.

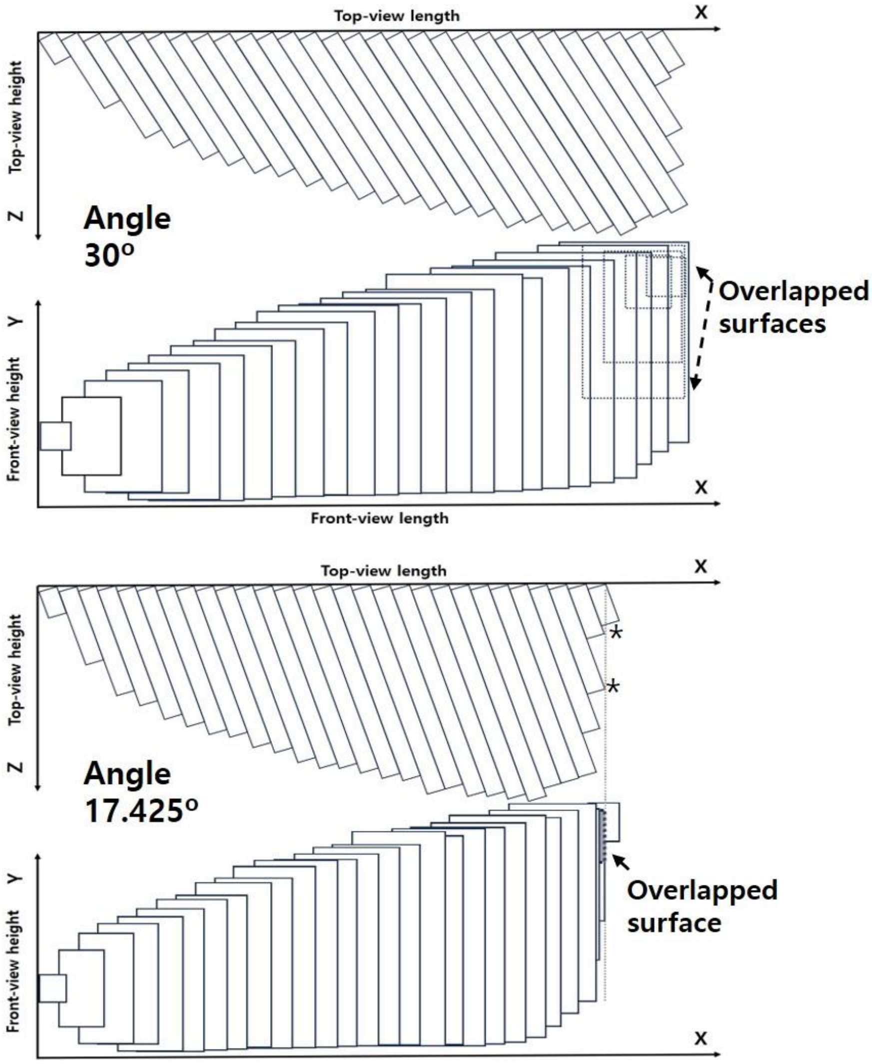

Figure 4: Case study of Example 1. Overlapping images of the cuboid sculpture (30º and 17.425º).

Segmented and Apparent Surface Area of Front View Dependent on Length Variations

The segmented surface area of the front-view in Figure 2 is summarized in Table 3. Each vertical segment may have overlapped surface areas. In this case, the apparent surface area of front view in each segment becomes the segmented surface area with the longest height. Ultimately, the total surface area of the front view can be expressed as follows.

$$Surface\ area\ of\ the\ front\ view = \sum_i L_i \times Max(H_i) \quad (7)$$

Segment	1	2	3	4	5	6	7
Surface areas of sub-segments	H1xL1	H1xL2					
		H2xL2*	H2xL3	H2xL4*	H2xL5*		
				H3xL4	H3xL5	H3xL6	
					H4xL5*	H4xL6*	H4xL7
Apparent surface area	H1xL1	H1xL2	H2xL3	H3xL4	H3xL5	H3xL6	H4xL7

Table 3. Segmentation of front view surface area for the Example 2 case study.
H is the height of each cuboid. L is the segmented length of
the angled stack of cuboids. * Symbolizes overlapped surface area.

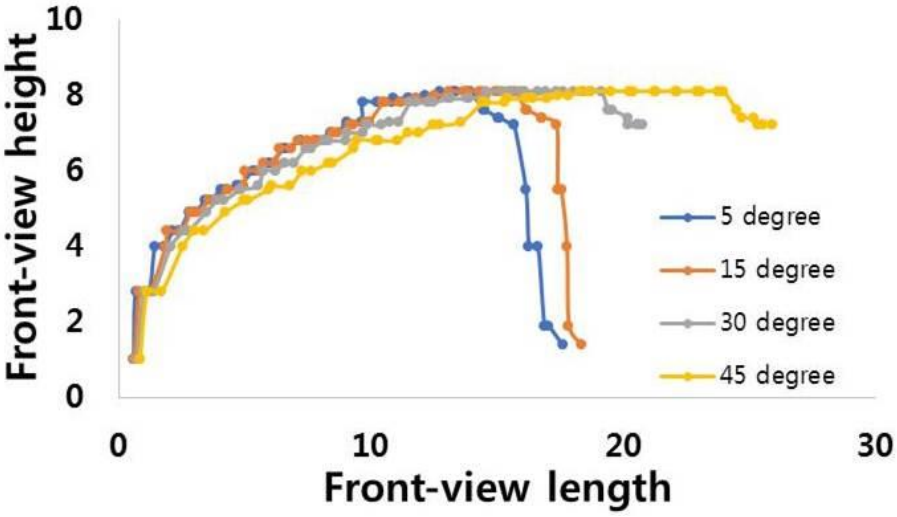

Figure 5: Case study of Example 1.
Variation of height of the front-view of the cuboid stack.

According to Eqn. 1, as the angle increases, the total length increases. And, Eqn. 7 tells us that the height at each segmented surface becomes the largest height among the segmented surfaces. Figure 5 shows the height of the segmented surfaces according to each angle as applied to the Example 1 case study. Therefore, the total surface inevitably increases as the angle increases. By applying Eqn. 7 to the Example 1 case study, the front-view surface area could be calculated according to each angle (Table 2). In addition, the simulation images could be compared, after incorporating the top-view surface according to angle variation into 2D painting (Figure 6). As shown in Figure 6, it can be seen that the harmony with the 2D painting perspective appears differently depending on the angle variation of the multilayer cuboid.

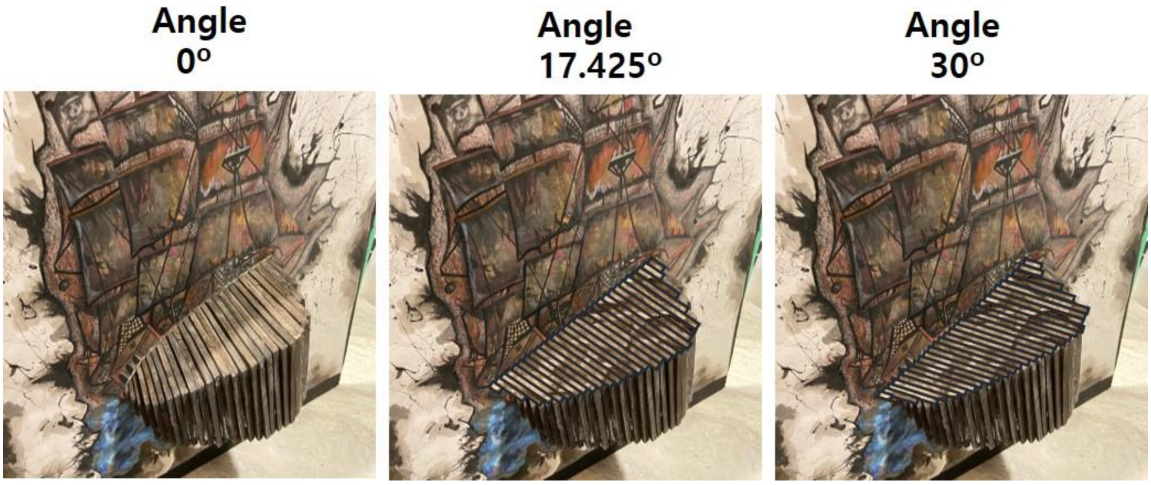

Figure 6: Comparative images of top-side view of the original painting depending on angle variation of the cuboid stack.

Conclusion

This study is about geometrical analysis according to angle variation in one-directional multilayer cuboid onto a painting. It was proven that as the angle increases, the visual length and surface area of the multilayer stack increase. Meanwhile, overlapping is a phenomenon caused by differences in the relative lengths of each cuboid, and in this study, the mathematical conditions under which such overlapping can occur were confirmed. Additionally, it was ascertained that as the angle increases, the possibility of overlapping also increases.

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Contact email: claireshin.college@gmail.com